Physics: Test n° 3

January 24th 2020. Duration: 3 hours

Instructions:

Presentation *and* ***redaction*** *of your paper will influence the marking. A calculator and a 2-sided hand written formula sheet are allowed. All three parts of this exam are independent and can be worked upon in any order.*

Part I
Measurement of charges by use of capacitive or inductive effect (approx. 7 points)

The goal of this 1st part is to show how very simple structures such as the ones seen during tutorials (a cylindrical capacitor or a torus) can be implemented as sensors. We will consider two types of detection: capacitive and inductive detection. In both cases the output of the sensor is a voltage which is related to the physical quantity to be measured. Both exercices in this part are completely independant.

Divergence and rotational are given in cylindrical coordinates:

$$div\ \vec{E} = \frac{1}{r}\frac{\partial}{\partial r}(r\ E_r) + \frac{1}{r}\frac{\partial E_\theta}{\partial \theta} + \frac{\partial E_z}{\partial z}$$

$$r\vec{o}t\ \vec{E} = \left(\frac{1}{r}\frac{\partial E_z}{\partial \theta} - \frac{\partial E_\theta}{\partial z}\right)\vec{u_r} + \left(\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r}\right)\vec{u_\theta} + \left(\frac{1}{r}\frac{\partial}{\partial r}(r\ E_\theta) - \frac{1}{r}\frac{\partial E_r}{\partial \theta}\right)\vec{u_z}$$

1 Capacitive sensor

Let's consider a cylindrical capacitor (Figure 1) of height H, internal radius R_1, and external radius R_2. The space between the plates of the capacitor is filled with air of ***permittivity*** ϵ_0*. Regarding the topography of the electric field we consider the capacitor as being infinite (edge effects are neglected). A potential difference $U_0 = V(R_1) - V(R_2)$ is applied between the plates ($U_0 > 0$), consequently the capacitor bears the charge Q.* ***In the following all the calculations need to be detailed and justified.***

Question 1 : Make a sketch and determine the expression of U_0 as a function of Q, ϵ_0 and of the geometrical dimensions of the capacitor.

Question 2 : Deduce from it the capacitance of the capacitor.

Now the charged capacitor is isolated from the voltage source U_0*. This way, each plate of the capacitor is electrically isolated from the surroundings. We fill the space between the plates with a material of permittivity* ϵ_0 *that contains a* ***uniform charge distribution of volume density*** ρ*. Subsequently we measure a new voltage difference $U = V(R_1) - V(R_2)$ at the terminals of the capacitor.*

Question 3 : With the help of a detailed sketch and by using **Maxwell-Gauss equation imperatively in point form**, as well as an appropriate boundary condition prove that the electric field $\vec{E}$ between the plates can be expressed as: $\vec{E} = \frac{\rho r}{2\epsilon_0} + \frac{1}{r}(\frac{Q}{2\pi H\epsilon_0} - \frac{\rho R_1^2}{2\epsilon_0})\ \vec{u}_r$.

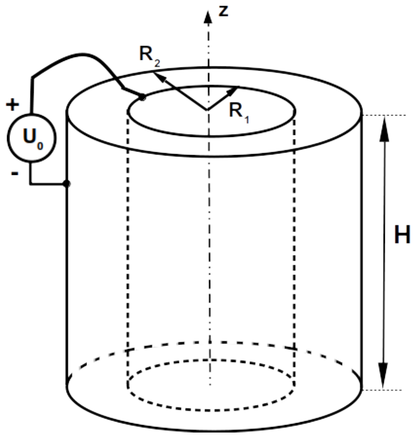

Figure 1: Cylindrical capacitor used to realize a sensor

Question 4 : What is the new potential difference U at the terminals of the cylindrical capacitor? The answer should be put it in the form: $U = U_0 + \Delta U$.

Question 5 : We assume that $R_2 = R_1 + e$, with $e \ll R_1$. Using this hypothesis, deduce an approximate expression of ΔU, as a function of ε_0, ρ and e.

It is given that for $a \ll 1$: $ln(1 + a) \simeq a - \frac{a^2}{2}$

Question 6 : Numerical application: Given that the sensor's electronics enables a measurement of at least 1 mV, and that the distance between the two plates is 1 mm, what is the smallest volumetric charge density that can be measured with this sensor? It is recalled that: $\epsilon_0 = \frac{1}{36\pi \cdot 10^9}\ \mathrm{F{\cdot}m^{-1}}$

2 Inductive sensor

The second type of sensor we want to study is constituted of a magnetic torus of absolute permeability μ, around which a winding comprising N turns is wound. The surface inside the torus (shaded surface in Figure 2 of normal unit vector $\vec{n}$ oriented along the (0z) axis) is denoted S. Moreover, the cross-section of the torus (located in a plane perpendicular to the plane of the figure) is a disk and is called S_c. The winding wound around the torus builds an electric circuit which is open. Consequently, no current flows through this circuit.

Through the surface S there is a flow of charged particles. They contain a volume charge density $\rho(t)$ which is uniform over the surface of the disk S, but may depend on time.

The velocity of the particles $\vec{v}(t)$ has the same orientation as $\vec{n}$ (perpendicular to the plane of the figure and towards the observer) and is assumed to be uniform all over the surface S.

Question 7 : Using dimensional analysis, show that the volume current density $\vec{j}$ can be written $\vec{j} = \rho\vec{v}$. Find the equivalent current I crossing the surface S.

Question 8 : Considering the flux of particules in the cylindrical tube of axis (Oz) and section S, calculate as a function of ρ, S, μ, R, and v the magnetic field $\vec{B}$ inside the torus at a distance R (corresponding to the average radius of the torus). You must **imperatively** use the integral form of Ampère's Law **justifying all steps**. You will consider that the quasi-static assumption applies here.

Question 9 : In the case where ρ **and/or** v **varies with time,** and assuming that the magnetic field $\vec{B}$ is **uniform** inside the torus and equal to the value found for $r = R$, deduce from the previous question the voltage e that could be measured accross the windind as a function of ρ, v, S, S_c μ,R, N.

Question 10 : Explain how this setup can be used as a sensor.

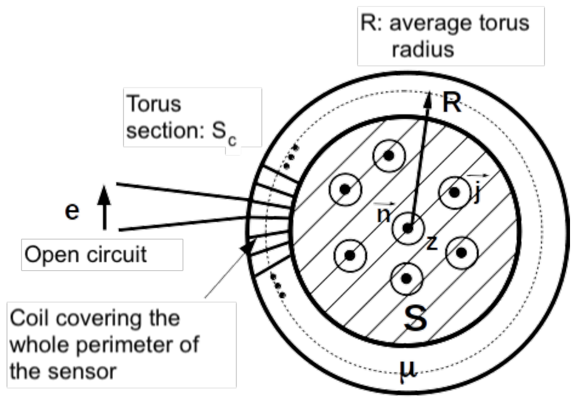

Figure 2: Inductive sensor crossed by a flow of charged particles.

Part II

Transmission of data from a weather balloon (approx. 9 points)

We use a balloon to measure the properties of air. The measurements from sensors on board of the balloon are transmitted in the form of an electromagnetic wave to a receiving antenna placed on the ground. The distance D balloon/antenna is fixed to $D = 20$ km. It is assumed that the waves propagate in air, assimilated to vacuum, at the speed $c = 3 \cdot 10^8$ m·s^{-1} and of frequency $f = 100$ MHz. The balloon is located as a point O, and the center of the antenna is called O'. The axis y is the axis (OO'), orientated from the balloon towards the antenna (see Figure 3).

Your reasonings must be supported by clear diagrams.

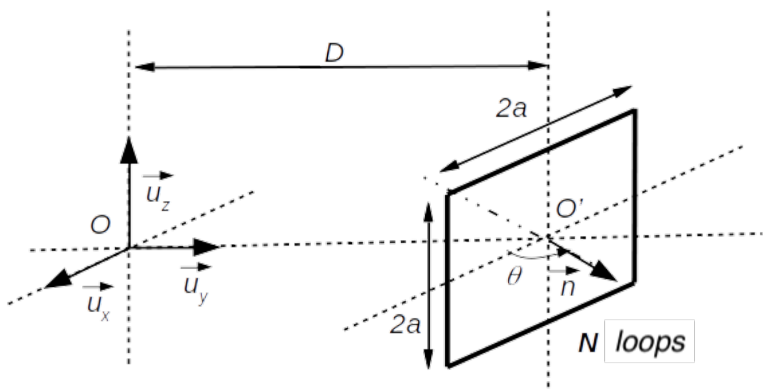

Figure 3: Antenna that receives waves emitted by the weather balloon

1 Preliminairy questions

Question 1 : Calculate the wavelength λ of the electromagnetic waves propagating in air.

Question 2 : Why is it necessary to take into account the propagation in this exercice?

Question 3 : What assumption from the text allows to consider that they are no energy losses in this media?

2 Characteristics of the electromagnetic wave

The electric field $\vec{E}$ emitted by the balloon is given by :

$$\vec{E} = E_0\ cos(\omega t - ky)\vec{u}_z$$

Question 4 : Why can we say it is a harmonic wave?

Question 5 : Give the **equation** of the wave surfaces and comment.

Question 6 : Is this wave uniform? Justify.

Question 7 : How is the electric component of this electromagnetic wave polarized?

Question 8 : In what direction (and orientation) does the wave propagate?

Question 9 : Justify (without demonstration) why the formula $\vec{B} = \frac{\vec{u}_y}{c} \wedge \vec{E}$ can be used for the calculation of the magnetic component $\vec{B}$.

We will admit that the electromagnetic wave keeps all these properties everywhere in space, also on the receiving antenna.

3 Time representation of the wave in y=0 and in y=D

Question 10 : Draw as precisely as possible the evolution of the amplitude of the wave in $y = 0$ between $t = 0$ (moment when the emission starts) and $t = 2T$ (T being the period of the wave).

Question 11 : At what time t_D does the wave reach the antenna in $y = D$?

Question 12 : Draw as precisely as possible the evolution of the amplitude of the wave in $y = D$ between t_D and $t_D + 2T$.

Question 13 : Lets assume that the signal is emitted during the time $4T$ starting from $t = 0$. Draw as a function of the position the shape of the wave at $t = t_D/2$. Indicate clearly the position of the forefront (beginning) and backfront (ending) of the wave.

4 Expression of the electric and magnetic fields on the y-axis, for a weather balloon transmitting since t = 0.

Figure 3 depicts a receiver antenna, whose non-negligible spatial dimensions can be represented as a square metallic frame of sides $2a$, composed of an N-turn winding. The center O' of the antenna is situated at a distance $OO' = D$ from the weather balloon. The antenna's spatial orientation is given by the reference unit vector $\vec{n}$, whose angle with the x-axis is θ.

Question 14 : Based on Figure 3, sketch the receiver antenna in the $(O', \vec{u}_x, \vec{u}_y)$ plane, considering that its width (given the N loops from which it is made of) is negligible which allows to consider it as a square of sides $2a$ with center at O'.

Question 15 : Using the formula from question 9, give the expression of the wave's $\vec{B}$ field for a position $M(x, y, z)$ (over the receiver antenna) assuming that the weather balloon transmits uninterruptedly since $t = 0$. You shall also give, as a function of θ, a and D, the extreme values of y and the instant t from which all the points of the antenna are affected by the electromagnetic wave.

5 Calculation of the induced emf in the receiver antenna

Question 16 : The vector $\vec{n}$ corresponds to which orientation (conventionally positive) of the current intensity i in the coil? A diagram must support your answer.

Question 17 : Give the expression of the elementary flux of $\vec{B}$, noted as $d\phi$, passing through an elementary antenna surface $d\vec{S}$ (for which you shall precise the orientation, without expressing dS). Deduce the value of θ for which $d\phi$ is maximal.

Question 18 : Using Figure 3 as a starting point, sketch a diagram representing the receiver antenna **for this particular value of** θ, as well as the magnetic field $\vec{B}$ and the vectors of the reference frame at point $O'(0, D, 0)$.

Question 19 : **In this configuration**, show that the expression of the flux of the magnetic field $\vec{B}$ through the N **spires** of the receiver antenna can be written as:

$$\phi = \frac{4aNE_0}{kc}\ sin(ka)\ cos(\omega t - kD)$$

We remind the trigonometric relation: $sin(p) - sin(q) = 2\ sin(\frac{p-q}{2})cos(\frac{p+q}{2})$

Question 20 : Deduce the expression of the emf e induced in the receiver antenna.

Question 21 : How much would the emf be if $a = \lambda$? Justify your result using a graph, showing how the norm of $\vec{B}$ varies on the antenna surface.

Question 22 : For what conditions on a is the l'amplitude of e maximal? The solution of the equation is not needed.

Question 23 : Write the approximated expression of e when $a \ll \lambda$. Was this result predictable?

Part III
Reception of information by the antenna (approx. 4 points)

The information signal $e(t)$ is obtained at the receiver antenna terminals, and then fed into an RLC circuit (see Figure 4). In order to extract the information sent by the weather balloon, we focus on the output voltage $s(t)$ obtained at the terminals of the capacitor C once a stable sinusoidal regime has been reached.

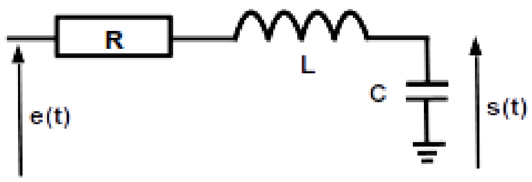

Figure 4: RLC receiver circuit

Question 24 : Express $\underline{s(t)}$ as a function of $\underline{e(t)}$ and of the complex impedances of the elements of this circuit. Deduce the complex transfer function $\underline{H} = \frac{\underline{s(t)}}{\underline{e(t)}}$ of this system. You shall provide this transfer function in the form:

$$\frac{1}{X + jY}.$$

Express the values of X and Y as a function of R, L, C and ω.

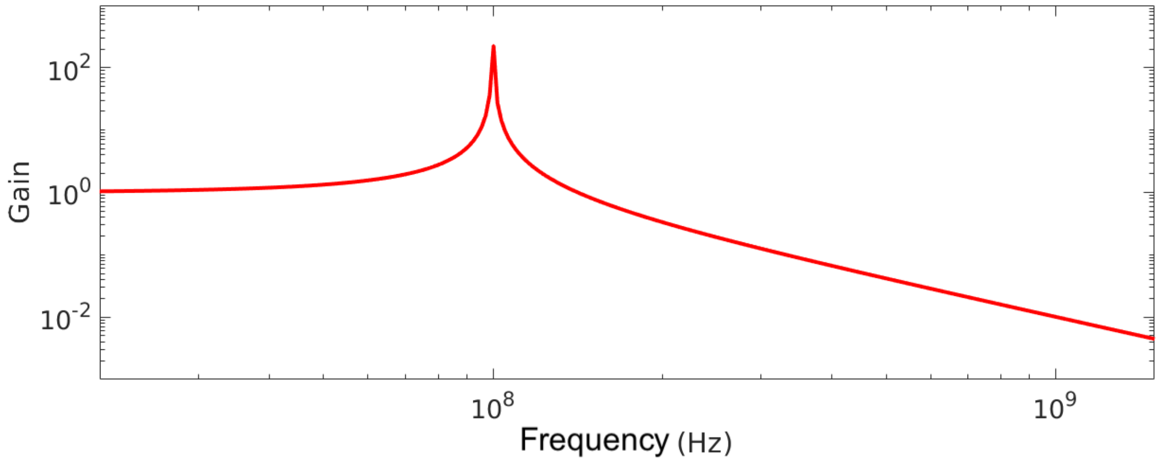

Figure 5: Modulus of $\underline{H}$ as a function of the frequency f

Question 25 : Using $\omega_0 = \dfrac{1}{\sqrt{LC}}$, express the modulus of $\underline{H}$ as a function of ω_0 and of the other electrical parameters of the system.

The modulus of $\underline{H}$ is given in Figure 5 for a limited frequency range between 20 MHz and 2 GHz.

Question 26 : Justify the limits of the modulus of $\underline{H}$ when $\omega \to 0$ and $\omega \to +\infty$.

Question 27 : What is the advantage of choosing this type of circuit to receive a 100 MHz wave? You shall justify your answer using values from the graph in Figure 5.

Question 28 : What is the phase shift φ_s of $s(t)$ with respect to $e(t)$ for an angular frequency of $\omega_0 = \dfrac{1}{\sqrt{LC}}$?

Question 29 : At this angular frequency (ω_0), which value of R provides a gain of 100?

Numerical data: *$C = 2.53$ pF and $L = 1$ μH.*