Physics Test n° 2

Monday 7th December 2020

Time Allowed: 1h30

Indicative mark scheme: Exercise 1 out of 10 points, exercise 2 out of 10 points. All documents and calculator allowed.

Not only your results, but above all your ability to clearly justify them and to analyse them critically will be evaluated. The mark scheme is given as a guide only.

The following self-certification text must be copied out and signed on your answer paper: *"I hereby declare that I will not cheat. Specifically, I will respect the instructions given on the test paper and not interact with anybody other than the teacher invigilating the test."*

Exercise 1 – Exercise related to the study of Capacitors (~10 points)

We will consider a parallel plate capacitor, of which the plates are discs with radius a separated by a distance ℓ. The medium separating the plates is air. We will neglect edge-effects.

Reminders

Maxwell-Ampère equation (in vacuum) : $\overrightarrow{rot}\left(\frac{\vec{B}}{\mu_0}\right) = \vec{J_c} + \vec{J_d} = \gamma\vec{E} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

Energy density associated with electric field $\vec{E}$: $w_e = \frac{\epsilon_0 E^2}{2}$ en J/m³

Permittivity of free space (air) : $\varepsilon_0 = {}^{1}/_{(36\,\pi\,10^9)}$ F/m (SI units)

Permeability of free space (air) : $\mu_0 = 4\,\pi\,10^{-7}$ H/m (SI units)

I. CAPACITOR IN ELECTROSTATIC EQUILIBRIUM

The capacitor is connected to a generator supplying a DC voltage U. The plates of the capacitor hold charge $+Q$ and $-Q$ *respectively.*

Determine the direction of $\vec{E}$ and show that its magnitude is constant between the plates (plot a graph).
Determine the relation between $\vec{E}$ and the surface charge density σ, then the total charge Q.
Deduce the expression for the capacitance C of the capacitor as a function of a, ℓ and other necessary constants.
Give the expression for the energy stored W_C in the capacitor as a function of U, C.
Show from this that we can find the expression for the electrostatic energy density.

II. STUDY OF CAPACITOR CHARGE CYCLE (rC CIRCUIT).

We charge the capacitor (from an initial discharged state) by connecting it to a generator supplying a DC voltage U through a switch that we close at a time $t = 0$. A transitory current $i(t)$ appears. The circuit connecting the generator and the capacitor has a resistance r.

Make a diagram of the circuit.
Determine the expression for the current $i(t)$ and make a graph of its form.
Calculate the energy dissipated by the Joule effect in the resistance of the circuit, given by $W_J = \int r\, i^2 dt$ as a function of U, C (taking into account that $i(t \to \infty) = 0$).
What is the energy supplied by the source, W_S, to charge the capacitor (to be established as a function of U, C).
Make an energy balance analysis (supplied, dissipated, stored) and comment it.

III. ELECTROMAGNETIC BEHAVIOUR OF THE CAPACITOR IN THE VARIABLE SUPPLY REGIME (sinusoidal in our case).

The capacitor is supplied by a sinusoidal voltage. The current crossing the wire connecting to the capacitor is

$I = I_0 cos(\omega t)$, and all associated values are sinusoidal.
The point form of the Maxwell-Ampère equation, given in the 'Reminders' box, reveals the presence of two types of volume current densities. We will study the relative contributions of the conduction current density ($\vec{J_c} = \gamma\vec{E}$) and the so-called displacement current density ($\vec{J_d} = \varepsilon_0 \frac{\partial\vec{E}}{\partial t}$).

III.1 Conductive part of the capacitor (plates)

Let us suppose that a sinusoidal electric field $\vec{E}$ of the form $E_m sin(\omega t)$ is present in the conductive part of the capacitor (plates).

Give the expression of the ratio of the current density ***amplitudes****:* $\alpha = \frac{\|\vec{J_c}\|}{\|\vec{J_d}\|}$ *as a function of* γ*,* ω *and other necessary constants.*
Numerical application: γ = 10^7 *S/m (conductor),* f = 10^6 *Hz.*
Calculate α *and comment the result.*
Calculate E_m *needed to obtain a current density amplitude of 1A/mm².*

III.2 In the space between the plates (filled with air: insulating medium).

Let's assume now that there is, between the plates, a sinusoidal electric field $\vec{E}$ of the form $\mathrm{E_0 sin(\omega t)}$.

What is the conduction current density $\vec{J_c}$ *between the plates of the capacitor?*

No charge carrier (electrons in the present case) goes across the space between the plates, however there is a current that crosses the capacitor in time-variable regime. The continuity of the current is due to the displacement current. That's what we are going to prove in the next-coming part.

We will assume that the boundary conditions and related properties remain identical to those seen in **electrostatics**, in particular regarding the charge distribution on the conducting plates and the value of the electric field inside the conducting plates.

Express the electric field $E(t)$ *between the plates as a function of the surface charge density* $\sigma(t)$ *and then as a function of the total charge* $Q(t)$*.*
Express the displacement current density j_D *between the plates and prove that we find again that* $i(t) = \frac{dQ}{dt}$*.*
Numerical application: γ = 10^7 *S/m (conductor plates),* f = 10^6 *Hz.*
Calculate E_0 *to obtain a current density of amplitude 1A/mm² and comment by comparing the result of question III.1.*

Exercise 2: Exercise related to the study of solenoids (~10 points)

The formulae related to the divergence and rotational in cylindrical coordinates are recalled below:

$$div\,\vec{A} = \frac{1}{r}\left(\frac{\partial(rA_r)}{\partial r} + \frac{\partial A_\theta}{\partial \theta} + \frac{\partial(rA_z)}{\partial z}\right)$$

$$\overrightarrow{rot}\,\vec{A} = \frac{1}{r}\left(\frac{\partial A_z}{\partial \theta} - \frac{\partial(rA_\theta)}{\partial z}\right)\overrightarrow{u_r} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\overrightarrow{u_\theta} + \frac{1}{r}\left(\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right)\overrightarrow{u_z}$$

We consider 2 identical solenoids having the same axis (Oz) and placed symmetrically with respect to the (xOy) plane) (figure 1). The solenoids are crossed by a current I which is time independent.

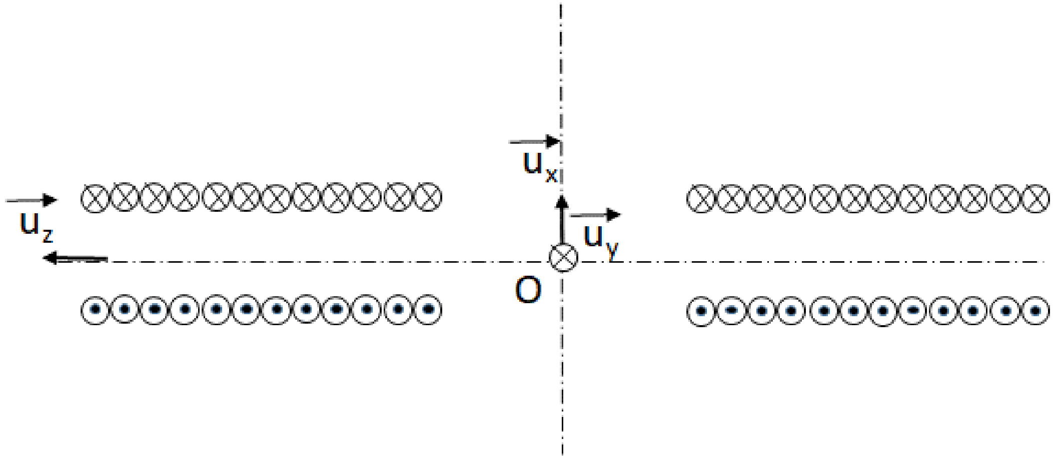

Figure 1: Sketch representing the solenoids

The field map representing the total magnetic field in the plane $(xOz)$is given in Figure 2. The x-axis is vertical, and the z-axis horizontal.

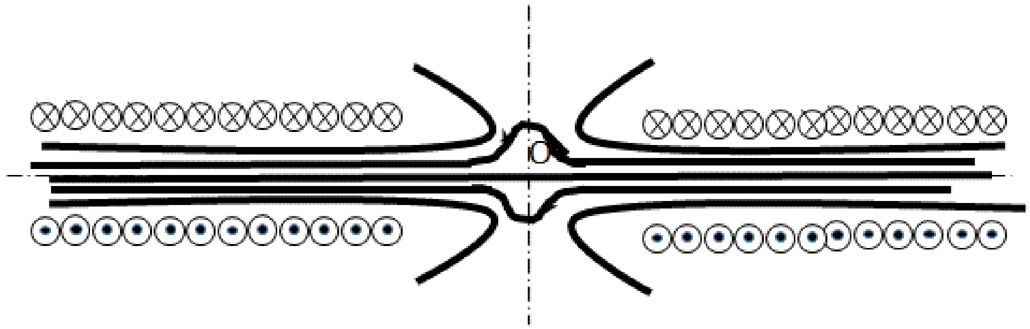

Figure 2: Sketch of the field lines of the magnetic field in the (xOz) plane

I. ANALYSIS OF THE MAGNETIC FIELD CREATED BY THE TWO SOLENOIDS

I.1 What is the orientation of the magnetic field along the field line going through point O?
I.2 At a given point M, represent the basis of cylindrical coordinates. Make use of the symmetries/antisymmetries and invariances of the current distribution in the basis of cylindrical coordinates (O, $\vec{u}_r$, $\vec{u}_\theta$, $\vec{u}_z$) to draw conclusions on the topography of $\overrightarrow{B}$.
I.3 What can we say about the field lines with respect to the (xOy) plane? Deduce that for the points belonging to that plane, in the basis of cylindrical coordinates (O, $\vec{u}_r$, $\vec{u}_\theta$, $\vec{u}_z$), the radial component of $\vec{B}$, denoted $B_r(r, z = 0)$ is nil.

We are now interested in the magnetic field inside the solenoids, far from the edges

I.4 By making use of the drawing representing the field lines, which component(s) can be considered as nil inside the solenoids?
I.5 We are now in the conditions mentioned in question 1.4

a) *Write Maxwell's equations in point form which are fulfilled by the magnetic field $\overrightarrow{B}$ inside the solenoids.*
b) *Deduce that the magnetic field $\overrightarrow{B}$ is uniform inside the solenoids. Its modulus will be denoted B_0.*

II. MEASUREMENT OF THE MAGNETIC FIELD AT THE POINT O

Now, we add a square loop of side $2a$, centered at O, and whose normal vector is aligned with the (Oz) axis. This loop is then connected to a voltmeter, in such a way as to not affect the field lines.

II.1 Both solenoids are powered by **a d.c. current** *I*

a) *In this configuration, what is the value of the voltage read on the voltmeter? (No calculation is required, but the answer must be properly justified)*
b) *Without changing the characteristics of the magnetic field, what should we do to the loop to for a potential difference to appear across its terminals? (No calculation is required, but the answer must be properly justified)*

Both solenoids are now powered by **a slowly time-varying current** $i(t)$.

II.2 Are we here in the general case of time-dependent regimes? If not, which approximation applies and why?

Now, let the $\overrightarrow{B}$ field on a point of the xOy plane be expressed as $\overrightarrow{B}(x, y, z = 0, t) = B_0(t) \exp\left(-\frac{x^2+y^2}{\alpha^2}\right)\overrightarrow{u_z}$

II.3

a) *Explain why an e.m.f. comes up in the loop.*
b) *Provide the literal expression of this e.m.f., carefully detailing every step of your reasoning and the choice of orientation you had to make. The integral that will be provided in the expression shall not be calculated.*
c) *To be able to complete the literal calculation, what hypothesis shall we make, and what does it imply qualitatively on the size of the loop? Provide the expression of the e.m.f. under this hypothesis.*
d) *How can this loop be used to measure the $\overrightarrow{B}$ field?*

III. EXPERIMENTAL ANALYSIS **N.B.: <u>this part requires no calculation</u>**

We allow two pendulums to oscillate (one at a time) in the xOy plane from the previous set-up, as shown in (Figure 3). The first one is made of copper, a conductive material, while the second one is made of PVC, an isolating material.

We measure the angle θ of the pendulum with respect to the vertical reference axis as a function of time in two distinct configurations: i) when the solenoids are not powered ($I = 0$); and ii) when they are powered by a continuous current I. The measurements for both configurations are shown in (Figure 4).

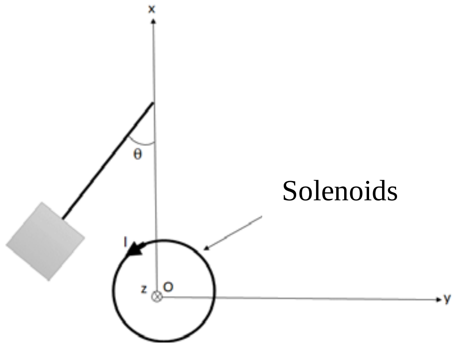

Figure 3: Pendulum oscillating in the (xOy) plane

III.1 With regard to the results, how do we justify that the friction is negligible?

III.2 Provide, in a few lines, your interpretation of the observed results

III.3 Which application illustrates this experiment?

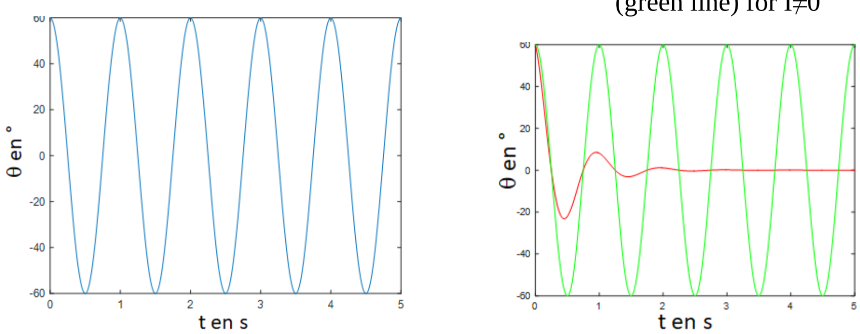

Figure 4 : Evolution of the angle θ as a function of time and of the nature of the pendulums' material depending on the current powering the solenoids.