# Physics Written Test n° 2

**Monday 6th December 2021** **Time Allowed: 1h30**

*Indicative mark scheme: exercise 1 out of 8 points, exercise 2 out of 12 points.*
*No documents allowed, calculators permitted.*

*Not only your results will be marked, but above all your capacity to clearly justify and analyse them in a critical manner will be evaluated. The mark scheme given above is purely indicative.*

## Exercise 1: Magnetostatics (~8 pts)

Let $\left(O,\vec{u}_x,\vec{u}_y,\vec{u}_z\right)$ be a direct orthonormal basis associated with the Cartesian coordinates $(x,y,z)$. We place our system in air with magnetic permeability $\mu_0$.

1. Question from class: be sure to be rigorous in your arguments and to schematically represent the system studied. The z = 0 plane contains an infinite sheet of surface current of density $\vec{k}=k\vec{u}_y$ where $k>0$. Determine the magnetic field created by this current distribution on either side of the sheet courant ($z>0$ and $z<0$).
2. We add at the plane $z=a>0$ a sheet of surface current of density $\vec{k'}=-k\vec{u}_y$. Make a sketch of this system in the *(xOz)* plane. Use the superposition principle and the result of the previous question to determine the total magnetic field created by the two current sheets in all space (apart from exactly on the 2 current sheets). Check that the boundary conditions at z = a, describing the discontinuity/continuity of the components of the magnetic field, are verified for the total magnetic field.
3. The following device is independent of those studied previously. We consider a solenoid with square cross-section. Its axis is parallel to the *(Oz)* axis and it has *n* turns of electrical wire per meter. The square cross-section of the solenoid has sides of length *a*, and its length is *h*, supposed to be much larger than *a*. When the wire hosts a current *I*, a magnetic field $\vec{B}=\mu_0 nI\vec{u}_z$ is generated in the median plane $z=\dfrac{h}{2}$ inside the solenoid.

   Make a detailed sketch of the device studied. Show that the magnetic field is identical elsewhere in the solenoid as long as edge-effects are neglected. Determine the self-inductance *L* of this circuit.

## Exercise 2: Conical Capacitor (~12 pts)

A list of formulas is available at the end of this exercise.
Consider two electrically conductive cones whose half-aperture angle at the apex are $\theta_1$ and $\theta_2$, respectively, and which are isolated electrically at the coordinate system's origin $O$. Both cones are surrounded by air, with permittivity $\varepsilon_0$. Let us denote $L$ the side length of both cones (also the radius of a containing sphere), as shown in the figure 1. These two conical conductors form the plates of a conical capacitor.
The axis of these two cones is oriented in the same direction of the unitary vector $\vec{u_z}$ and we denote $(r,\theta,\varphi)$ the spherical coordinates of a point $M$ in the space <u>between the two cones</u>. A potential difference of $U=V_2-V_1>0$ is maintained between both cones. We consider the cone 1 (whose apex half-aperture is $\theta_1$) to be at potential $V_1=0\text{V}$ and cone 2 (whose apex half-aperture is $\theta_2$) to be at potential $V_2>0$. Bord effects are to be neglected throughout this exercise.

1. Sketch the cut view of the device in the $(xOz)$ plane. Your sketch must contain the spherical coordinates and the local frame at a point $M$ of your choosing between the capacitor plates (conductors).
2. Using Maxwell's equations, show that the potential $V$ in the space in-between the capacitor plates verifies a point form equation (involving a differential operator)

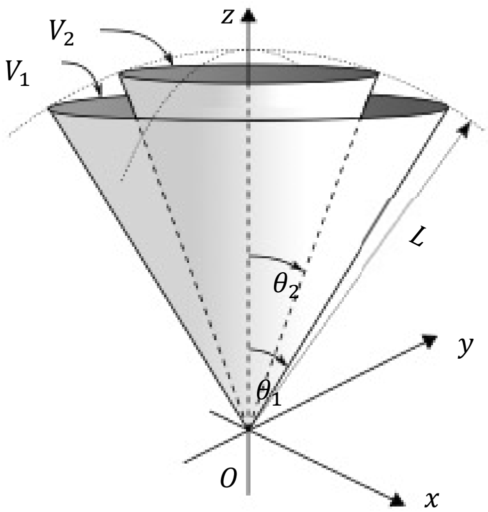


Figure 1: Layout of a conical capacitor of side *L*.

3. We assume that the potential *V* depends *only* on the spherical coordinate $\theta$.
   a) Explain why this hypothesis is coherent with the charge distribution.
   b) Calculate *V* as a function of the potential difference *U*, of the angle $\theta$ and of the angles $\theta_1$ and $\theta_2$.
   Indication :
   $\int \frac{d\theta}{sin(\theta)} = ln\left(tan\left(\frac{\theta}{2}\right)\right) + K$ (*K* constant)
4. a) Justify the shape of the field lines of the electric field $\vec{E}$ and represent them on a sketch.
   b) Deduce from the preceding question that the field is oriented along a unit vector $\vec{u}$ of the local spherical basis and indicate what is $\vec{u}$. Justify the orientation of $\vec{E}$.
5. Calculate the field $\vec{E}$ as a function of *r*, $\theta$ and $\vec{u}$. Prove that the field $\vec{E}$ can be expressed in the form $\vec{E} = f\left(\theta_1, \theta_2\right)\frac{U}{r\sin(\theta)}\vec{u}$ where *f* is a function of the parameters $\theta_1$ and $\theta_2$ to be determined.
6. Determine the surface charge density $\sigma_2$ on the cone of half-aperture angle $\theta_2$ as well as the total charge $Q_2$ carried by this plate as a function of *f* and of the angles $\theta_1$ or $\theta_2$. Deduce from it the capacitance *C* of the conical capacitor as a function of *f* and of the angles $\theta_1$ or $\theta_2$. Subsequently express C as a function of the sole data of the exercise ($\theta_1$ and $\theta_2$).
7. Calculate again the capacitance *C* of the capacitor by making use of the concept of density of electrostatic energy stored. First express *C* as a function of *f* and of the angles $\theta_1$ or $\theta_2$. Subsequently express C as a function of the sole data of the exercise ($\theta_1$ and $\theta_2$).
8. Numerical application: calculate the capacitance of the conical capacitor if $\theta_1 = \frac{\pi}{2}$, $\theta_2 = \frac{\pi}{3}$, $L = 100$ µm, and $\varepsilon_0 \cong 8{,}9 \cdot 10^{-12}$ F/m.

**Operator expressions in spherical coordinates:**

$$\overrightarrow{grad} f = \frac{\partial f}{\partial r}\vec{u}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\vec{u}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\vec{u}_\varphi$$

$$div\vec{F} = \frac{1}{r^2}\frac{\partial\left(r^2 F_r\right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial\left(\sin\theta F_\theta\right)}{\partial\theta}\vec{u}_\theta + \frac{1}{r\sin\theta}\frac{\partial F_\varphi}{\partial\varphi}$$

$$div\left(\overrightarrow{grad} f\right) = \Delta f = \frac{1}{r^2\sin(\theta)}\left[\frac{\partial}{\partial r}\left(\mathrm{r}^2\sin\theta\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{\partial}{\partial\varphi}\left(\frac{1}{\sin\theta}\frac{\partial f}{\partial\varphi}\right)\right]$$

**Volume element in spherical coordinates:** *(dr)(rd$\theta$)(rsin$\theta$ d$\varphi$)*

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CORRECTION IE2 – PHYSIQUE

<table>
<tr><td></td><td>Exercice 1</td><td></td><td>8</td></tr>
<tr><td>1</td><td>Figure correctement complétée (axes du repère, représentation de $\vec{k}$ et contour d'Ampère orienté, etc)<br>Nappe de courant invariante par toute translation dans $xOy$ : $\vec{B}(z)$<br>Le plan $yMz$ est plan de symétrie des courants, donc $\vec{B}(M) = B_x\vec{u}_x$<br>$xOy$ est plan de symétrie de $\vec{k}$ et $B_x(-z) = -B_x(z)$<br>Soit le contour d'Ampère rectangle $C = A_1A_2A_3A_4A_1$ dans le plan $xMz$, de largeur $L$ et passant par $A_1(x-\frac{L}{2}, y, z)$ et $A_3(x+\frac{L}{2}, y, -z)$, orienté par $\vec{u}_y$ :<br>$\int_{A_1}^{A_2}\vec{B}\cdot\vec{d\ell} = B_x(z)L$, $\int_{A_2}^{A_3}\vec{B}\cdot\vec{d\ell} = 0 = \int_{A_4}^{A_1}\vec{B}\cdot\vec{d\ell}$,<br>$\int_{A_3}^{A_4}\vec{B}\cdot\vec{d\ell} = -B_x(-z)L = B_x(z)L$ avec $B_x(z)$ impair.<br>$I_{tot} = \int_{x-\frac{a}{2}}^{x+\frac{a}{2}}\vec{k}\cdot d\ell\vec{u}_y = kL$. Finalement le Th d'Ampère $\int_C \frac{\vec{B}}{\mu_0}\cdot\vec{d\ell} = I_{tot}$ conduit à :<br>$B_x(z>0) = \mu_0\frac{k}{2}$ et $B_x(z'<0) = -B_x(-z'>0) = -\mu_0\frac{k}{2}$</td><td>0,5<br><br>0,25<br>0,5<br><br>0,25<br><br>Total de 2 (0,5 contour, 0,5 orientation cohérente, 0,5 flux de k, 0,5 résultat)<br>Accepter une démo par les relations locales + passage</td><td>3,5</td></tr>
<tr><td>2</td><td><table>
<tr><td></td><td>$z<0$</td><td>$0<z<a$</td><td>$a<z$</td></tr>
<tr><td>$\vec{B}$</td><td>$-\mu_0\frac{k}{2}\vec{u}_x$</td><td>$\mu_0\frac{k}{2}\vec{u}_x$</td><td>$\mu_0\frac{k}{2}\vec{u}_x$</td></tr>
<tr><td>$\vec{B}'$</td><td>$-\mu_0\frac{k'}{2}\vec{u}_x = \mu_0\frac{k}{2}\vec{u}_x$</td><td>$\mu_0\frac{k}{2}\vec{u}_x$</td><td>$\mu_0\frac{k'}{2}\vec{u}_x = -\mu_0\frac{k}{2}\vec{u}_x$</td></tr>
<tr><td>$\vec{B}_{tot}$</td><td>$\vec{B}+\vec{B}' = \vec{0}$</td><td>$\mu_0 k\vec{u}_x$</td><td>$\vec{0}$</td></tr>
</table></td><td>Figure 0,5<br><br>Tableau 1,5 (0,25 par case juste pour B' et Btot)</td><td>2</td></tr>
<tr><td></td><td>Définir milieux 1 et 2 (ici z < a milieu 1 et z > a milieu 2, d'où la normale $\vec{u_z}$)<br>$\Delta\vec{B}_\perp = \vec{0}$ vérifiée puisque $\vec{B}_\perp = \vec{0}$<br>$\Delta\left(\frac{\vec{B}_\parallel}{\mu_0}\right) = \vec{k}'\wedge\vec{u}_z$ : vérifié puisque $\vec{k}'\wedge\vec{u}_z = -k\vec{u}_x$ et $\Delta\left(\frac{\vec{B}_\parallel}{\mu_0}\right) = \vec{0} - k\vec{u}_x$</td><td>0,25<br>0,25<br><br>0,5</td><td>1,0</td></tr>
<tr><td>3</td><td>Figure complétée avec représentation des données<br>Si on néglige les effets de bord : on peut considérer le solénoïde infini et il est alors invariant par translation parallèle à $Oz$, d'où $\vec{B}$ indépendant de $z$ et $\vec{B}(z) = \vec{B}\left(\frac{h}{2}\right)$<br>Solénoïde de longueur $h$ donc de $N = nh$ spires ; comme de plus $\vec{B}$ considéré invariant, le flux à travers chaque spire du solénoïde considéré comme constant et $\Phi_{tot} = N\varphi_1$ avec $\varphi_1 = \iint_{1\text{spire}} \vec{B}\cdot\vec{n}dS = a^2B_z$ le flux à travers 1 spire.<br>Il vient $L = \frac{\Phi_{tot}}{I} = \frac{Na^2B_z}{I} = \mu_0 n^2a^2h$</td><td>0,25<br><br>0,25<br><br><br>1 (0,5 si oubli N spires)</td><td>1,5</td></tr>
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|---|---|---|---|
| | Exercice 2 | | 12 |
| 1 | Coupe, coordonnées & repère local | 1 | 1 |
| 2 | Equation de Maxwell-Gauss $div(\vec{E})=0$<br>$\vec{E}=-\vec{grad}(V)$<br>d'où $\Delta V=0$ | 0,25 + 0,25<br><br>0,5 | 1,0 |
| 3 | a) Cette hypothèse est cohérente avec les données du problème car<br>1/ invariance par rotation φ<br>2/ les cônes conducteurs $\theta=\theta_1$ et $\theta=\theta_2$ sont des équipotentielles<br>b) $\Delta V=\frac{1}{r^2\sin(\theta)}\left[\frac{\partial}{\partial\theta}(\sin(\theta)\frac{\partial V}{\partial\theta})\right]=0$<br>Donc $\left[\frac{d}{d\theta}(\sin(\theta)\frac{dV}{d\theta})\right]=0$<br>$\frac{dV}{d\theta}=\frac{A}{\sin(\theta)}$<br>$V=A\ln(\tan(\frac{\theta}{2}))+B$ avec A,B constantes d'intégration<br>Conditions aux limites<br>$V(\theta_1)=A\ln(\tan(\frac{\theta_1}{2}))+B=0$<br>$V(\theta_2)=A\ln(\tan(\frac{\theta_2}{2}))+B=V_2=U$<br>$V=\frac{U\ln(\frac{\tan(\theta/2)}{\tan(\theta_1/2)})}{\ln(\frac{\tan(\theta_2/2)}{\tan(\theta_1/2)})}$ | <br>0,25<br>0.5<br><br><br><br>0,5<br>0,25 (cte B)<br><br>0,25<br><br>0,25<br><br>0,5 | 2,5 |
| 4 | a) Les lignes de champ sont perpendiculaires aux équipotentielles<br>donc les lignes de champ sont des arcs de cercle centrés sur le point O<br>b) On en déduit $\vec{E}=E\vec{u_\theta}$ avec E > 0 car $\vec{E}$ descend les potentiels | 0,5<br>0,5<br>0,25<br>0,25 | 1,5 |
| 5 | $\vec{E}=-\vec{grad}(V)$<br>$\vec{E}=-\frac{1}{r}\frac{dV}{d\theta}\vec{u_\theta}$<br>$\vec{E}=-\frac{U}{r\sin(\theta)\ln(\frac{\tan(\theta_2/2)}{\tan(\theta_1/2)})}\vec{u_\theta}$<br>$f=-\frac{1}{ln(\frac{tan(\theta_2/2)}{tan(\theta_1/2)})}$ | <br><br>0,25<br><br>0,25 | 0,5 |

| | | | |
|---|---|---|---|
| 6 | Le champ est nul dans le conducteur donc les conditions de passage du champ électrostatique (ou bien énoncé du théorème de Coulomb) donnent<br>$\sigma_2 = \varepsilon_0 \vec{E}(r,\theta_2).\vec{u_\theta}$<br>$\sigma_2 = -\dfrac{\varepsilon_0 U}{r\sin(\theta_2)\ln(\dfrac{\tan(\theta_2/2)}{\tan(\theta_1/2)})} = \dfrac{\varepsilon_0 K}{r\sin(\theta_2)}$<br>Elément de surface sur le cône $dS = 2\pi r\sin(\theta_2)dr$<br>Charge totale<br>$Q = \int_0^L -\dfrac{2\pi U\varepsilon_0 dr}{\ln(\dfrac{\tan(\theta_2/2)}{\tan(\theta_1/2)})} = \dfrac{2\pi UL\varepsilon_0}{\ln(\dfrac{\tan(\theta_1/2)}{\tan(\theta_2/2)})}$<br>On en déduit C<br>$C = \dfrac{2\pi L\varepsilon_0}{\ln(\dfrac{\tan(\theta_1/2)}{\tan(\theta_2/2)})}$ | (accepter avec f non explicitée)<br>0,5<br>0,5<br>0,5<br>0,5 | 2,0 |
| 6 | Densité volumique d'énergie électrostatique<br>$w = \dfrac{\varepsilon_0 E^2}{2}$<br>$w = \dfrac{\varepsilon_0 K^2}{2(r\sin(\theta))^2}$<br>Elément de volume $dV = 2\pi r^2\sin(\theta)drd\theta$<br>Energie électrostatique totale<br>$W = \iint_{0\le r\le L,\theta_1\le\theta\le\theta_2} \dfrac{\varepsilon_0 K^2}{2(r\sin(\theta))^2} 2\pi r^2\sin(\theta)drd\theta$<br>$W = \pi\varepsilon_0 LK^2\ln(\dfrac{\tan(\theta_2/2)}{\tan(\theta_1/2)})$<br>$W = \dfrac{CU^2}{2}$ on retrouve la même expression pour C | (accepter avec f non explicitée)<br>0,5<br>0,75<br>0,75<br>0,5 | 2,5 |
| 7 | C=4.3 $10^{-3}$ pF | 1 | 1 |