1ST PHYSICS EXAM - 3RD SEMESTER - SECOND YEAR

The two exercises are totally independent. Any result that is not justified will not be taken into account. Care should be taken to ensure the presentation and legibility of the exam.

Useful formulas:

Gradient in cylindrical coordinates:
$\vec{\nabla} U = \vec{\text{grad}}(U) = \frac{\partial U}{\partial r}\vec{u}_r + \frac{1}{r}\frac{\partial U}{\partial \theta}\vec{u}_\theta + \frac{\partial U}{\partial z}\vec{u}_z$

Divergence in cylindrical coordinates:
$\vec{\nabla} \cdot \vec{E} = \text{div}(\vec{E}) = \frac{1}{r}\frac{\partial}{\partial r}(r\ E_r) + \frac{1}{r}\frac{\partial E_\theta}{\partial \theta} + \frac{\partial E_z}{\partial z}$

Rotational (curl) in cylindrical coordinates:
$\vec{\nabla} \times \vec{A} = \vec{\text{rot}}(\vec{A}) = \left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\right)\vec{u_r} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\vec{u_\theta} + \left(\frac{1}{r}\frac{\partial}{\partial r}(r\ A_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial \theta}\right)\vec{u_z}$

1 Study of a vector field

Consider a vector field $\vec{F}$ defined in a given region of space by its rotational (curl) vector and divergence in a cylindrical reference frame $(\vec{u}_r, \vec{u}_\theta, \vec{u}_z)$:

$$\vec{\text{rot}}(\vec{F}) = \vec{\nabla} \times \vec{F} = \alpha\, \vec{u}_z$$

$$\text{div}(\vec{F}) = \vec{\nabla} \cdot \vec{F} = 0$$

Where α is a positive constant. We also know that this vector field is invariant by rotation of the angle θ and by translation along direction z. We also know that $\vec{F}$ is zero for $r = 0$.

Question 1-1 : *Find the complete expression of $\vec{F}$ in the space considered.*

From the vector field $\vec{F}$, we define another vector field $\vec{A}$ such that :

$$\vec{F} = \vec{\nabla} \times \vec{A} = \vec{\text{rot}}(\vec{A})$$

This second vector field $\vec{A}$ is defined such that $\vec{A}(0, \theta, z) = k\vec{u}_z$ (k is also a positive constant). $\vec{A}$ does not depend on z or θ and has a single non-zero component along $\vec{u}_z$.

Question 1-2 : *Give the complete expression of $\vec{A}$ in the space considered.*

Question 1-3 : *Does the $\vec{F}$ field derive from a scalar potential? Justify.*

Question 1-4 : *Is the $\vec{F}$ field flux conservative?*

We now consider a closed contour $ABCD$ consisting of 4 branches forming a rectangle of height H and width d. The orientation of this contour is shown in the figure 1. This contour delimits a surface S which **must be correctly oriented**.

Question 1-5 : *Calculate the flux of $\vec{F}$ through the surface S, using a direct flux calculation.*

Question 1-6 : *Compute the circulation of $\vec{A}$ along the Γ contour **using a direct calculation**. How could this result be found otherwise?*

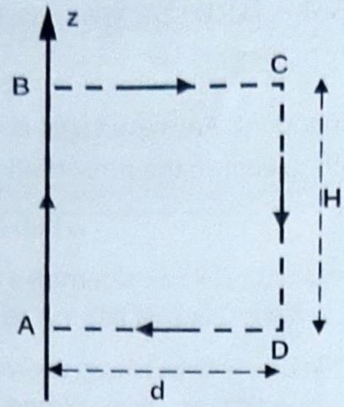

Figure 1: *Calcul du flux de $\vec{F}$ et de la circulation de $\vec{A}$*

2 Principle of a capacitive level sensor

The aim of this subject is to study the principle of a two-wire capacitive sensor. This sensor purpose is to measure the level in a tank filled with a liquid of dielectric permittivity ϵ.

2.1 Preliminary section

Initially, we consider a cylinder (C), with axis Oz vertical and unit vector $\vec{u}_z$, radius R, length L, charged in volume with a uniform charge density ρ ($\rho > 0$). For the analysis of symmetries and invariances, we'll assume that the cylinder is infinite along the Oz axis, even though the length L of the cylinder is finite. We'll also assume that the dielectric permittivity is equal to ϵ_0 inside and outside the cylinder in this part of the exercise.

Question 2-1 : *Make a sketch of the device and give the expression for the total charge Q in the cylinder.*

Question 2-2 : *Define, in the most appropriate coordinate system and at any point M in space, the topography of the electric field, that is, which components of the electric field are non-zero as well as the variables they depend on.*

Question 2-3 : *Establish the expression of the electric field produced at any point M in space (inside or outside the cylinder) as a function of ρ, R, r and ϵ_0. Represent how the electric field modulus evolves as a function of the distance of point M from the wire's axis Oz.*

The radius R tends towards a value so small that the cylinder (C) is reduced to a wire of very small cross-section (close to zero), carrying a linear charge density λ.

Question 2-4 : *Establish the relationship linking ρ and λ, and then prove that the expression for the electric field at a point M located* ***outside*** *from the wire can be written as:*

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0 r}\,\vec{u}$$

where $\vec{u}$ is a unit vector to be specified.
This result (even if not proved) can be used in the rest of the exercise.

2.2 Principle of a two-wire level sensor

We now consider two conductive wires of length L, radius δ (quasi-zero, but non-zero), separated by a distance d such that $d \ll L$, and considered infinite along the Oz axis (see figure 2). They are respectively charged with linear charge density $+\lambda$ and $-\lambda$ and are placed at potentials V_1 and V_2, respectively, such that $V_2 < V_1$. The two wires are in electrostatic equilibrium and will be assumed to be in the case where electrostatic induction is the strongest (total influence). They are initially located in air of dielectric permittivity ϵ_0.

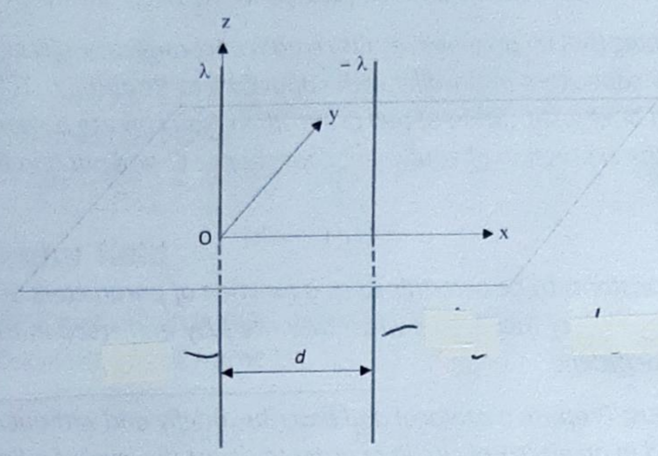

Figure 2: *Sketch of a two-wire capacitive level sensor*

Question 2-5 : *Provide a detailed analysis of the symmetries/anti-symmetries and invariances of the system formed by these 2 linear charge distributions.*
What can be concluded about the total electric field (the sum of the individual electric fields created by each wire) contained in the (xOz) plane? And in the (xOy) plane?
Draw the vectors for the total electric fields at the points M_1 $(0, d, 0)$, M_2 $(d, d, 0)$ and M_3 $(d, -d, 0)$ located in the (xOy) plane. ***Justify your sketch of these vectors of electric field.*** *An approximate scale of these vectors is sufficient for the sketch. The drawing must be reported in the given appendix and handed back with your papers.*

Question 2-6 : *Using the expression of electric field created by an infinite wire with linear charge density $+\lambda$, give, as a function of x, d, λ, ϵ_0 and a suitable unit vector, the expression of the electric field at any point M of abscissa x ($\delta < x < d - \delta$) located on the segment that connects the 2 wires in the (xOy) plane (see figure in the appendix). The electric field is assumed to be nil inside the wires.*

Question 2-7 : *Calculate the potential difference $U = V_1 - V_2$ between the two wires as a function of d, δ, λ and ϵ_0. Put the expression of $U = V_1 - V_2$ in the form:*

$$V_1 - V_2 = a\frac{\lambda}{\epsilon_0}\ln\left(\frac{b}{c}\right)$$

where a, b and c are constants to be determined. ***In the following, you will use this expression with a, b and c.***

Question 2-8 : *The elementary charge dQ contained in an elementary length dz of* ~~*each*~~ He *wire is given as follows:*

$$dQ = dC \times U$$

where dC is the elementary capacitance associated to the elementary length dz. Deduce the expression of dC as a function of dz, ϵ_0, a, b and c.

We consider now that the 2 wires are immersed to a height of $h < L$ in a liquid of permittivity ϵ. The non-immersed parts of the wires are still located in the air of permittivity ϵ_0.

Question 2-9 : *Assuming that all previous results remain unchanged, briefly explain that everything is as if there were two capacitors in parallel with capacitances C1 and C2.* ~~*Taking into account all previous results, give a brief justification of this proposition*~~ *(you can use a sketch to help justify this proposition). Deduce the expression of equivalent capacitance C and put it in the following form:*

$$C = C_0(1 + \alpha h)$$

where C_0 and α are constants to be determined as a function of parameters of the problem.
Note : C_0 is the capacitance of the 2 wires when they are fully immersed in the air (only) and α is called the sensitivity coefficient.

Question 2-10 : *Bonus: Propose a protocol and describe, briefly and without any calculation, how this device can be used in an electrical circuit in order to detect the level of a liquid.*