Physics: Test n°2

Monday December 18, 2023 Duration : 1h30

Exercise 1 (~11 pts)

We consider a cylinder of radius R, assumed to have an infinite length in the z-direction. It is crossed by a volume current density $\vec{j} = j_0 \frac{r}{R} \vec{u}_z$ (for $r \in [0; R]$) defined in a cylindrical coordinate system ($\vec{u_r}, \vec{u_\theta}, \vec{u_z}$), as shown in Figure 1. This cylinder is made of a magnetic material of permeability μ and it is placed in air of permeability μ_0.

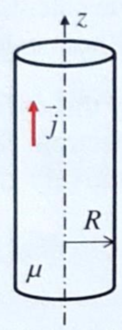

Figure 1

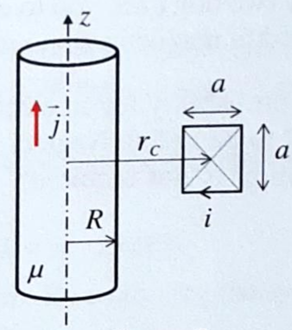

Figure 2

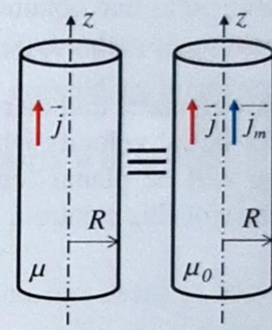

Figure 3

Gradient of a scalar vector field U in cylindrical coordinates:

$$\vec{\nabla} U = \overrightarrow{grad}\, U = \frac{\partial U}{\partial r}\vec{u_r} + \frac{1}{r}\frac{\partial U}{\partial \theta}\vec{u_\theta} + \frac{\partial U}{\partial z}\vec{u_z}$$

Rotational (curl) of a vector field $\vec{B}$ in cylindrical coordinates:

$$\vec{\nabla} \wedge \vec{B} = \overrightarrow{rot}(\vec{B}) = \frac{1}{r}\left(\frac{\partial B_z}{\partial \theta} - \frac{\partial (rB_\theta)}{\partial z}\right)\vec{u_r} + \left(\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r}\right)\vec{u_\theta} + \left(\frac{1}{r}\frac{\partial (r\, B_\theta)}{\partial r} - \frac{1}{r}\frac{\partial B_r}{\partial \theta}\right)\vec{u_z}$$

1. After having analysed the topography of the magnetic field $\vec{B}$, recall a useful boundary condition that applies at the interface $r=R$. Using local relations establish then with a detailed calculation the expression of the magnetic field $\vec{B}$ at any point in space.
2. Express the current I crossing the cylinder as a function of the data of the exercise. Deduce then the expression of the magnetic field $\vec{B}$ (which may be called $\vec{B}_{ext}$) outside the cylinder as a function of I.
3. Someone moves a square current loop of side a crossed by the current i close to the cylinder until its centre is located at the distance r_C (see Figure 2). Determine the mutual inductance denoted M between the cylindrical wire and the square current loop.
4. This square current loop is now considered small ($a \ll r_C$). Give the expression of its magnetic moment and determine the wrench of the magnetic forces (resulting force and resulting moment) acting on it. Comment.

Questions 5 and 6 are unrelated and can be answered independently.

5. This cylinder, made of a magnetic materiel of permeability μ through which the current density $\vec{j} = j_0 \frac{r}{R} \vec{u}_z$ flows, is equivalent to a **non-magnetic cylinder (permeability μ_0)** through which the global current density $\vec{j} + \vec{j_m}$ flows, where $\vec{j_m}$ represents a current density related to the magnetization of magnetic materiel (Figure 3), $\vec{j}$ remaing the same as before. We assume that $\vec{j_m}$ is of the form $\vec{j_m} = K\, j_0 \frac{r}{R} \vec{u}_z$ where K is an unknown constant.

   a) Express K ,which comes up in the expression of $\vec{j_m}$, as a function of μ_0 and μ so that the magnetic field $\vec{B}$ inside of the cylinder is identical to the magnetic field calculated in the question 1.

   b) Is this representation with this volume current $\vec{j_m}$ sufficient to ensure that the external field $\vec{B}$ is identical to that obtained in question 1 (we don't ask you to carry out the complete calculations, but to show that another type of current linked to magnetization must be considered without calculating it).

6. We now consider that there is a non-zero probability for an electron to be emitted by this cylindrical wire with an initial velocity which is assumed to be radial. Neglect the weight of the electron. Show that its motion will be planar (complete calculation is not required). Do a sketch, qualitatively depicting the trajectory of this electron.

Exercice 2 (~3 pts)

Consider an infinite wire through which a current I flows along its z-axis as shown in figure 4. The magnetic field $\vec{B}$ created by this current at any point of the space M (r, θ, z) is given to be:

$$\vec{B} = \frac{\mu_0 I}{2\pi r} \vec{u_\theta}$$

A segment of wire AB of length ℓ, through which a current i is flowing, is approached perpendicularly to the infinite wire (Figure 4).

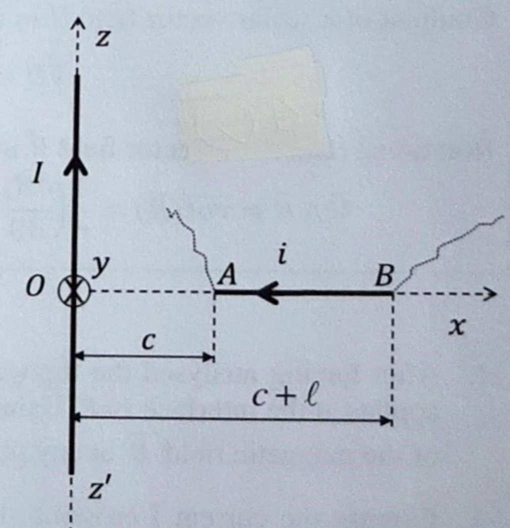

Figure 4

1. Give the expression of the elementary Laplace force $\overrightarrow{dF}$ resulting from the interaction between $\vec{B}$ and i at a point M belonging to AB. Represent each physical quantities in this expression. Calculate the Laplace force $\vec{F}$.

2. The elementary moment $d\Gamma_{Oy}$ of $\overrightarrow{dF}$ with respect to the Oy axis, at point M, is given by: $d\Gamma_{Oy} = (\overrightarrow{OM} \wedge \overrightarrow{dF}).\vec{u}_y$. Calculate Γ_{Oy}.

3. Analyze and comment the movement of the segment AB.

Exercice 3 (~6 pts)

Consider a ball of radius R placed at the origin of an orthonormal reference (O,x,y,z). This ball is a magnetic materiel characterized by a magnetization vector $\vec{M} = M\vec{u_z}$ that create a magnetic field $\vec{B}$ (Figure 5). We use a **spherical base system** $(\vec{u_r}, \vec{u_\theta}, \vec{u_\varphi})$ (Figure 6).

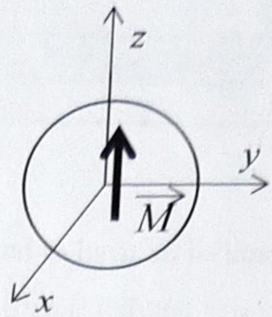

Figure 5

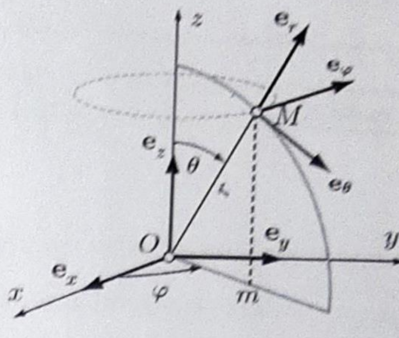

Figure 6

In order to calculate the magnetic field $\vec{B}$ created by this magnetized ball, the theory of the magnetostatics of magnetic media allows us to replace this ball by a sphere of permeability μ_0 crossed by a surface current density $\vec{k}$ such that $\vec{k} = \vec{M} \wedge \vec{n}$, where $\vec{n}$ is the unit normal vector outgoing the surface of sphere.

1. With the help of a sketch, show that $\vec{k} = M \cdot \sin(\theta)\vec{u_\varphi}$
2. Sketch some current lines on the sphere. By analyzing the symmetry of the current density, show that the magnetic field $\vec{B}$ can be written as: $\vec{B} = B_r(\mathrm{r},\theta)\vec{u_r} + B_\theta(\mathrm{r},\theta)\vec{u_\theta}$.
3. Recall the 2 Maxwell's equations verified by the magnetic field $\vec{B}$. Using the expressions of the operators given below, develop the partial differential equation fulfilled by $\vec{B}$ outside or inside of the surface of the sphere which is crossed by the surface current $\vec{k}$.
4. The medium inside of the surface of the sphere ($r < R$) is called medium 1 and the medium outside of the surface of the sphere ($r > R$) is called medium 2. We suppose that the solutions of the partial differential equation given in 3) are:

For medium 1: $B_{r,1}(\mathrm{r},\theta) = \dfrac{2\mu_0 M}{3}\cos(\theta)$, $B_{\theta,1}(\mathrm{r},\theta) = -\dfrac{2\mu_0 M}{3}\sin(\theta)$

For medium 2: $B_{r,2}(\mathrm{r},\theta) = \dfrac{2D_2}{r^3}\cos(\theta)$, $B_{\theta,2}(\mathrm{r},\theta) = \dfrac{D_2}{r^3}\sin(\theta)$,

where D_2 is a characteristic constant of the medium 2.

Using a boundary condition at the surface of the sphere (r = R), determine D_2.

Rotational (curl) in spherical coordinates:

$$\vec{rot}(\vec{B}) = \left\{\frac{1}{r\sin(\theta)}\left[\frac{\partial(\sin(\theta)B_\varphi)}{\partial\theta} - \frac{\partial B_\theta}{\partial\varphi}\right]\right\}\vec{u_r} + \left\{\frac{1}{r}\left[\frac{1}{\sin(\theta)}\frac{\partial B_r}{\partial\varphi} - \frac{\partial(rB_\varphi)}{\partial r}\right]\right\}\vec{u_\theta} + \left\{\frac{1}{r}\left[\frac{\partial(rB_\theta)}{\partial r} - \frac{\partial(B_r)}{\partial\theta}\right]\right\}\vec{u_\varphi}$$

Divergence in spherical coordinates:

$$div(\vec{B}) = \frac{1}{r^2}\frac{\partial(r^2 B_r)}{\partial r} + \frac{1}{r\sin(\theta)}\frac{\partial(\sin(\theta)B_\theta)}{\partial\theta} + \frac{1}{r\sin(\theta)}\frac{\partial B_\varphi}{\partial\varphi}$$