INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

FIMI, SCAN 1A *24nd of May 2024*

Mathematics Written Examination # 5

Duration: 1 hour 30 minutes. *All documents and electronic devices are prohibited.*

- *No documents, no calculators, no cell phones or electronic devices allowed.*
- *Take a deep breath before starting (everything is going to be ok !) and read entirely the exam before starting.*[0].
- *All exercises are independent, you can do them in the order that you'd like.*
- *Please start an exercise at the top of a page (for readability).*
- *Number single pages, or simply the booklets (copies doubles) if multiple: for example 1/3, 2/3, 3/3*
- *All your answers must be fully (but concisely) justified, unless noted otherwise.*
- *Redaction and presentation matter ! For instance, write full sentences and make sure your 'x' and 'n' can be distinguished.*
- ***Respecting all of the above is part of the exam grade (0.5 points).** Provided rubric is indicative (changes may occur).*

Exercise 1 ($\sim$ 5 points) Warm-ups

The following questions are independent. You are expected to provide some justification for those questions, especially for the True/False questions.

1. True or False : A polynomial $P \in \mathbb{R}[X]$ of degree four has either two inflection points or none.

2. True or False : the set $F = \{f \in \mathcal{F}(\mathbb{R}, \mathbb{R}) \mid \lim_{x \to +\infty} f(x) = 0\}$ is a vector subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

3. True or False : the family $\mathcal{F} = (X-1;\ X^2-1;\ (X-1)^2)$ is a basis of the vector space G defined by $G = \{P \in \mathbb{R}_2[x] \mid P(1) = 0\}$.

4. Find $a, b, c \in \mathbb{R}$ such that :
$$\begin{pmatrix} a & 1 & 0 \\ 2 & b & 1 \\ 1 & 1 & 3 \end{pmatrix} \times \begin{pmatrix} 3 \\ 1 \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

5. Which of the following statements are valid ?
   (a) $\ln(t) \underset{0}{=} t - \frac{t^2}{2} + o(t^2)$ ✗
   (b) $e^t \underset{0}{=} 1 + t + \frac{t^2}{2} + o(t^2)$ ✓
   (c) $e^{1+t} \underset{0}{\sim} 1 + (1+t)$ ✓
   (d) $\ln(t) \underset{1}{\sim} t$ ✓
   (e) $e^t \underset{0}{\sim} 1 + t - \frac{t^2}{2}$ ✗
   (f) $\ln(1+t^2) \underset{0}{=} t^2 + o(t^3)$ ✓

Exercise 2 ($\sim$ 7 points)

Let f be the function defined by :

$$f(x) = \begin{cases} 3 + 2x^2 \ln\left(1 + \frac{1}{x}\right) & \text{if } x > 0 \\ 3 \text{ if } x = 0 \end{cases}.$$

1. Using Taylor Series Expansions, determine the limit of f at $+\infty$, and whether the curve of f has an asymptote or not at $+\infty$.
   If that's the case, determine the equation of the asymptote and the relative positions of the curve and the asymptote near $+\infty$.
2. Show that f is differentiable at 0, and give the value of $f'(0)$. (Note that $1 + \frac{1}{x} = \frac{x+1}{x}$.)
3. (a) For $x > 0$, calculate $f'(x)$.
   (b) Using the Mean Value Theorem, show that :
   $$\forall t > 0, \ln(1+t) > \frac{t}{t+1}.$$
   (c) Deduce, with detailed reasoning, that f is a bijection from $I = [0, +\infty[$ to an interval J to be determined.
   (d) Study the differentiability of f^{-1} on J.
4. (a) Let $n \in \mathbb{N}$. Justify that from a certain rank n_0 to be determined, the equation $f(x) = \sqrt{n}$ has a unique solution (no need to calculate it). We denote this solution by x_n. Thus, a sequence $(x_n)_{n \geq n_0}$ is defined.
   (b) Determine the monotonicity of the sequence $(x_n)_{n \geq n_0}$.
   (c) Show that $x_n \underset{n \to +\infty}{\sim} \alpha n^\beta$ where α and β are real numbers to be determined.

Exercise 3 ($\sim$ 8 points)

In this problem, we are interested in the function f defined for all $x \in \mathbb{R}^*$ by

$$f(x) = \frac{x}{2\tanh(x/2)} = \frac{x \cdot \cosh(x/2)}{2\sinh(x/2)}.$$

We denote C_f as the graph of f.

1. Compute the second-order Taylor expansion of f at 0.
2. Deduce that f can be extended to a continuous and differentiable function at 0. We will still denote f as its extension.
3. Provide the equation of the tangent to the graph of f at 0. How is the graph of f positioned with respect to its tangent at 0 (in a small neighborhood of 0)?

We now admit that f is C^∞ on $\mathbb{R}$.

5. What is $f''(0)$?
6. What is the parity of f?
7. What can we deduce about $f'''(0)$ and, more generally, about the successive derivatives of f at 0?
8. It is given that $f'''(x) < 0$ for all $x \in]0; +\infty[$. Determine, with justification, the sign of $f'''(x)$ for $x \in]-\infty; 0]$.
9. Let P_2 be the second-order Taylor polynomial (the osculating parabola) of f at zero. Using the previous question and Taylor-Lagrange, formula determine the sign of $f(x) - P_2(x)$ on $\mathbb{R}^*$.
10. What can we deduce about $f^{(4)}(0)$? Justify your answer.

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0. **Draw a cube next to your name on the first page once this is done.**