Exercise 1 (4 pts)

1.	$\det(C) = \det(A) \times \det(B)$. Since matrices A and B are triangular, their determinant equals the product of their diagonal elements. Hence, $\det(C) = -6 \times 1 = -6$.	
2. (a)	$\begin{vmatrix} a-2 & 2 & -1 \\ 2 & a & 2 \\ 2a & 2a+2 & a+1 \end{vmatrix} \underset{C_1 \leftarrow C_1 - C_3}{=} \begin{vmatrix} a-1 & 2 & -1 \\ 0 & a & 2 \\ a-1 & 2a+2 & a+1 \end{vmatrix} \underset{L_3 \leftarrow L_3 - L_1}{=} (a-1) \begin{vmatrix} 1 & 2 & -1 \\ 0 & a & 2 \\ 0 & 2a & a+2 \end{vmatrix}$ Then, expanding along the first column, $\det(M_a) = (a-1)(a^2 + 2a - 4a) = a(a-1)(a-2)$.	
(b)	M_a is invertible $\iff \det(M_a) \neq 0 \iff a \in \mathbb{R} \setminus \{0;1;2\}$.	
3. (a)	$A \times \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$ so $f(u) = 2u$ with $u \neq \vec{0}$. u is therefore an eigenvector of f associated with the eigenvalue $\lambda = 2$.	
(b)	Let γ be the third eigenvalue of A. $\text{trace}(A) = 6 = \lambda + \mu + \gamma$, hence $\gamma = 3$ and the characteristic polynomial of A is therefore $-(X-1)(X-2)(X-3)$.	

Exercise 2 (4.75 pts)

1.	f is differentiable on $\mathbb{R}_+$, and for all $x \geqslant 0$, $f'(x) = -x\mathrm{e}^{-x} \leqslant 0$, so f is decreasing on $\mathbb{R}_+$. Furthermore, $f([0,1]) = [f(1), f(0)] = \left[\frac{2}{\mathrm{e}}, 1\right] \subset [0,1]$ so the interval $[0,1]$ is indeed stable under f .	
2.	$\forall x \in \mathbb{R}_+$, $h'(x) = -x\mathrm{e}^{-x} - 1 \leqslant 0$, and since h' only vanishes at 0, it is therefore strictly decreasing and continuous on $\mathbb{R}_+$. h thus defines a bijection from $\mathbb{R}_+$ to $h(\mathbb{R}_+) =]\lim_{+\infty} h, h(0)] =]-\infty, 1]$. Thus, there exists a unique real number $\alpha \geqslant 0$ such that $h(\alpha) = 0$, i.e., such that $f(\alpha) = \alpha$. Moreover, $h(1) = \frac{2}{\mathrm{e}} - 1 < 0$ hence $\alpha < 1$.	
3.	Since f is decreasing on $[0,1]$, $0 < \alpha < 1 \implies f(1) < f(\alpha) < f(0) \implies \frac{2}{\mathrm{e}} < \alpha < 1$.	
4.	We have $(f \circ f)(\alpha) = f(f(\alpha)) = f(\alpha) = \alpha$, so α is a fixed point of $f \circ f$. Since it is assumed that $f \circ f$ has only one fixed point on $[0, +\infty[$, it must necessarily be α.	
5.	• First, since $u_0 \in [0,1]$ and $[0,1]$ is stable under f, we can deduce that for all $n \in \mathbb{N}$, $u_n \in [0,1]$ (and thus (u_n) is bounded) . • Next, since f is decreasing on $[0,1]$, we know that the sequence (u_n) is not monotonic but the sequences $(v_n) = (u_{2n})$ and $(w_n) = (u_{2n+1})$ are both monotonic . • Being bounded (as subsequences of (u_n)), the sequences (v_n) and (w_n) are therefore convergent , and since $v_{n+1} = (f \circ f)(v_n)$ and $w_{n+1} = (f \circ f)(w_n)$, by the continuity of $f \circ f$ their limit must necessarily be the unique fixed point of $f \circ f$ on $[0,1]$, i.e., α (from Part I) . • Finally, since (u_{2n}) and (u_{2n+1}) both converge to α, we can deduce that (u_n) converges to α .	

Exercise 3

Part I (3.75 pts)

1.	$\sum_{k=n}^{2n-1}(x_{k+1}-x_k)=\sum_{k=n}^{2n-1}x_k-\sum_{k=n}^{2n-1}x_k=\sum_{k=n+1}^{2n}x_k-\sum_{k=n}^{2n-1}x_k=x_{2n}-x_n.$	
2.	From property (P), we have for all $n\geqslant n_0$, $$x_{2n}-x_n\geqslant\sum_{k=n}^{2n-1}\frac{\alpha}{k}\geqslant\sum_{k=n}^{2n-1}\frac{\alpha}{2n-1}.$$ Now, $\sum_{k=n}^{2n-1}\frac{\alpha}{2n-1}=(2n-1-n+1)\times\frac{\alpha}{2n-1}=\frac{\alpha n}{2n-1}\geqslant\frac{\alpha n}{2n}$. Hence, we have for all $n\geqslant n_0$, $x_{2n}-x_n\geqslant\frac{\alpha}{2}$.	
3.	First, property (P) implies that the sequence (x_n) is increasing starting from rank n_0 . Thus, (x_n) has a limit $\ell\in\mathbb{R}\cup\{+\infty\}$. However, if $\ell\in\mathbb{R}$, then $\lim_{n\to+\infty}(x_{2n}-x_n)=\ell-\ell=0$, which is impossible according to the inequality from the previous question .	

Part II (7.5 pts)

1.	For $x\in[0\,,1]$, $f'(x)=1-2x$. f is therefore strictly increasing on $\left[0\,,\frac{1}{2}\right]$ and strictly decreasing on $\left[\frac{1}{2}\,,1\right]$.	
2.	Let $n\in\mathbb{N}$, $f\left(\frac{1}{n+1}\right)-\frac{1}{n+2}=\frac{1}{n+1}-\frac{1}{(n+1)^2}-\frac{1}{n+2}=[...]=\frac{-1}{(n+1)^2(n+2)}\leq 0$	
3.	We prove by induction on $\mathbb{N}$ that $0<u_0<\frac{1}{n+1}$. $\underline{n=0:}$ we have $0<u_0=\frac{1}{2}<1$. $\underline{n=1:}$ we have $0<u_1=\frac{1}{4}<\frac{1}{2}$. $\underline{n\to n+1:}$ Assume that for some $n\in\mathbb{N}^*$, we have $0<u_n<\frac{1}{n+1}\leq\frac{1}{2}$ (since $n\geq 1$). Since f is strictly increasing on $\left[0\,,\frac{1}{2}\right]$, we have $f(0)<f(u_n)<f\left(\frac{1}{n+1}\right)$. From the previous question, we deduce that $0<u_{n+1}<\frac{1}{n+2}$. By induction, we have $0<u_n<\frac{1}{n+1}$ for all n in $\mathbb{N}$. Note : We need $n\in\mathbb{N}^*$ to prove the induction, so we must initialize the induction proof at $n=1$.	
4.	Let $n\in\mathbb{N}$, $v_{n+1}-v_n=(n+1)u_{n+1}-nu_n=(n+1)(u_n-u_n^2)-nu_n=u_n(1-(n+1)u_n)$. From question 3, $u_n>0$ and $1-(n+1)u_n>0$. Therefore, $v_{n+1}-v_n>0$.	
5.	From question 3, for all $n\in\mathbb{N}$, $0<v_n<\frac{n}{n+1}<1$. The sequence (v_n) is therefore increasing, bounded above by 1, and hence (v_n) is convergent to $\ell\in[0\,,1]$.	
6.	$v_1=u_1=\frac{1}{4}>0$. And (v_n) is increasing, so $\ell\geq\frac{1}{4}>0$.	
7.	Using the computation from question 4 : For all $n\in\mathbb{N}$, $w_n=n(v_{n+1}-v_n)=n(u_n(1-(n+1)u_n))=v_n(1-v_n-u_n)$. From question 3, $\lim_{n\to+\infty}u_n=0$ (by bounding) , and from question 5, $\lim_{n\to+\infty}u_n=\ell$. Thus, we have $\lim_{n\to+\infty}w_n=\ell(1-\ell)$.	
8.	Assume by contradiction that $\ell\neq 1$. We have already shown that $\ell\neq 0$. Hence, we deduce that $\ell'=\ell(1-\ell)>0$. We have $\lim_{n\to+\infty}w_n=\ell'$. By the definition of the limit, for sufficiently large n, $w_n\in\left[\ell'-\frac{1}{2}\ell'\,,\ell'+\frac{1}{2}\ell'\right]$. Taking $\alpha=\frac{\ell'}{2}$, we have $w_n\geq\alpha$ for sufficiently large n. Since $w_n=n(v_{n+1}-v_n)$, there exists n_0 such that for all $n\geq n_0$, $v_{n+1}-v_n\geq\frac{\alpha}{n}$. From part I, $\lim_{n\to+\infty}v_n=+\infty$, which contradicts question 3. Thus, we deduce that $\ell=1$. Finally, $u_n=\frac{v_n}{n}\underset{n\to+\infty}{\sim}\frac{1}{n}$.	