Exam n° 3 – 1 hour 30 minutes

- No documents, no calculators, no cell phones or electronic devices allowed.
- Take a deep breath before starting (everything is going to be ok!) and read entirely the exam before starting.⁰.
- All exercises are independent, you can do them in the order that you'd like.
- Please start an exercise at the top of a page (for readability).
- Number single pages, or simply the booklets (*copies doubles*) if multiple : for example 1/3, 2/3, 3/3
- All your answers must be fully (but concisely) justified, unless noted otherwise.
- Redaction and presentation matter! For instance, write full sentences and make sure your 'x' and 'n' can be distinguished.
- **Respecting all of the above is part of the exam grade (0.75 points).** Provided rubric is indicative (changes may occur).

Warm-up exercises (9 points)

You are expected to provide some steps for those exercises. Little partial credit will be given for just writing the answer.

Exercise 1. Sense or no sense, that is the question. Justify briefly why the following statements are meaningful or meaningless.

1. Sense or No Sense : *The function* $f : \mathbb{R}^2 \to \mathbb{R}$ *admits an image of dimension* 2 *so it is bijective.*
2. Sense or No Sense : *Consider the linear application* $p : E \to E$ *such that* $p \circ (-p) = -p$. *Then* p *is a projection.*
3. Sense or No Sense : *The rank of the kernel is linearly independent so it is injective.*
4. Sense or No Sense : $x = o(1)$ *at* $x = 0$, *but* $1 = o(x)$ *at* $x = \infty$.

Exercise 2. Consider $s = (1,0,0,1)$, $t = (0,1,1,1)$, $u = (0,-1,-1,1)$, $v = (2,0,0,0)$, and $B = (s,t,u,v)$ a family of $\mathbb{R}^4$.

1. Show that B is linearly dependent and provide its rank.
2. Complete the linearly independent subfamily to form a basis of $\mathbb{R}^4$.

Exercise 3. Consider $F = \text{Span}(e_1, e_2)$, $G = \text{Span}(e_4 - e_1, e_3)$ two vector subspaces of $\mathbb{R}^4$, expressed using the canonical basis of $\mathbb{R}^4$.

1. Provide the dimension of F, G.
2. Compute $F \cap G$. Are F and G supplementary?
3. Deduce a relation between F, G, and $\mathbb{R}^4$.

Exercise 4. Provide an equivalent at $x = 0$ of the following function (justify your steps) and deduce the limit.

$$f(x) = \frac{\sin(2x)\ln(1+x) + x\sin(2x)\ln(1+x)}{x(x+1)}$$

Exercise 5. Find the following limit (justify your result).

$$\lim_{x \to +\infty} \left(\frac{x^2 - 1}{x^2} \right)^x$$

Linear applications (5 points)

Exercise 6.
Consider the linear application $f : \mathbb{R}^3 \to \mathbb{R}^4$ such that $f(x, y, z) = (x + y, -x - y, 3x + 3y, 2z)$.

1. Write the associated matrix M relative to the canonical basis of $\mathbb{R}^3$ and $\mathbb{R}^4$.
2. Without computing neither the kernel or the image, determine the dimension of one of them (state explicitly which one you pick), and **deduce** the dimension of the other. Justify your answer.
3. Can f be injective? Surjective? Bijective? Justify each answer.
4. Provide a basis K of $\ker(f)$, and a basis I of $\text{Im}(f)$. We expect proper justification here.
5. Complete K with vectors to form a basis B of $\mathbb{R}^3$, and write the associated matrix M' relative to B and the canonical basis of $\mathbb{R}^4$.

Limits and Comparisons (5.25 points)

Exercise 7.
Consider $f(x) = \dfrac{\cos(x)(\cos(x) - 1) + \sin^2(x)}{x(1 - e^{3x})}$.

1. Determine the domain of definition of f.
2. Provide an equivalent of f at 0 and deduce the limit when $x \to 0$.
3. Can f be extended by continuity to $\mathbb{R}$? Justify. If yes, provide the expression of $\tilde{f}$, the extension.
4. Behavior at $+\infty$:
   (a) Find g such that $f(x) \underset{+\infty}{=} O(g(x))$.
   (b) Compute the limit of g at $+\infty$.
   (c) Conclude whether f admits an asymptote and $+\infty$ (if yes, specify what kind).
5. Behavior at $-\infty$:
   (a) Find h such that $f(x) \underset{-\infty}{=} O(h(x))$.
   (b) Compute the limit of h at $-\infty$.
   (c) Conclude whether f admits an asymptote and $-\infty$ (if yes, specify what kind).

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0. Draw a flower next to your name on the first page once this is done.