INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

FIMI, SCAN 1A *17th of June 2024*

Mathematics Written Examination # 6

Duration: 3 hours. *All documents and electronic devices are prohibited.*

- *No documents, no calculators, no cell phones or electronic devices allowed.*
- *Take a deep breath before starting (everything is going to be ok !) and read entirely the exam before starting.*[0]
- *All exercises are independent, you can do them in the order that you'd like.*
- *Please start an exercise at the top of a page (for readability).*
- *Number single pages, or simply the booklets (copies doubles) if multiple: for example 1/3, 2/3, 3/3*
- *All your answers must be fully (but concisely) justified, unless noted otherwise.*
- *Redaction and presentation matter ! For instance, write full sentences and make sure your 'x' and 'n' can be distinguished.*
- ***Respecting all of the above is part of the exam grade (0.5 points).*** *Provided rubric is indicative (changes may occur).*

Exercise 1 ($\sim$ 1 points) Let f be the function defined on $(0,+\infty)$ by $f(x)=-(x+2)e^{-\frac{1}{x}}+2$.

1. Determine the real numbers a, b, and c such that $f(x) \underset{+\infty}{=} ax+b+\dfrac{c}{x}+o\left(\dfrac{1}{x}\right)$.

2. Interpret graphically in terms of asymptote and relative position.

Exercise 2 ($\sim$ 2 points)

We consider the function f defined on $I=\left[0,\frac{\pi}{2}\right]$ by $f(x)=\cos x+\sin x$.

1. Study the variations of f on I.

2. (a) Let x be a real number in the interval $\left(\frac{\pi}{4},\frac{\pi}{2}\right]$. Write the Taylor-Lagrange formula for f of order $n=3$ on the interval $J=\left[\frac{\pi}{4},x\right]$ to express $f(x)$.

   (b) Let $x\in\left(\frac{\pi}{4},\frac{\pi}{2}\right]$. Bound the error made when using the polynomial of $\mathbb{R}_3[X]$ from the previous formula as an approximate value of $f(x)$ when $\left|x-\frac{\pi}{4}\right|<\frac{1}{10}$.
   Is this approximate value an overestimate or an underestimate?

Exercise 3 (~ 5 points)

We define the function f on $\mathbb{R}$ by :

$$f(x) = \begin{cases} 1 + x\dfrac{\operatorname{ch} x - 1}{\operatorname{sh} x} & \text{if } x \in \mathbb{R}^* \\ 1 & \text{if } x = 0 \end{cases}.$$

We recall the following relation :

$$\forall x \in \mathbb{R},\ (\operatorname{ch} x)^2 - (\operatorname{sh} x)^2 = 1.$$

Let C_f be the representative curve of f.

1. **Preliminary :** Calculate the Taylor series expansion to order 3 at zero for $\varphi : x \longmapsto \dfrac{1-x}{1+x}$.
2. (a) Calculate the Taylor series expansion of order 2 at zero for the function f.

   (b) What properties of f and its graphical representation C_f near zero can be deduced?
3. (a) Show that, on $\mathbb{R}^*$, the function f is differentiable and that its derivative has the same sign as $\operatorname{sh}(x) + x$.

   (b) Establish the variations of f on $\mathbb{R}$. (The limits at $-\infty$ and $+\infty$ are not required.)
4. (a) Determine an equivalent of f at $+\infty$ of the form $f(x) \underset{+\infty}{\sim} mx^p$ where m and p are two real numbers to be determined.

   (b) Can we deduce the existence of an asymptote to the curve C_f near $+\infty$ without additional calculations?
5. (a) Express $\alpha(x) = \dfrac{\operatorname{ch} x - 1}{\operatorname{sh} x}$ as a function of $s = e^{-x}$. Then perform the Taylor series expansion to order 1 at zero for this expression in the variable s.

   (b) Show that there exist three real numbers a, b, and c such that $f(x) \underset{+\infty}{=} ax + b + cxe^{-x} + o\left(xe^{-x}\right)$.
6. (a) What can be deduced for the curve C_f near $+\infty$?

   (b) Can anything be deduced from the previous question for the curve C_f near $-\infty$?

---

0. Replace "car" by "bicycle" on the last instruction of this exam.

Exercise 4 (~ 6 points)

We consider the $\mathbb{R}$-vector space $\mathbb{R}_2[X]$. We define the application on $\mathbb{R}_2[X]$ by :

$$f(P) = (X+1).P' + P\,.$$

(where P' is the derivative polynomial of P).

Let $\mathcal{B}_0$ be the canonical basis of $\mathbb{R}_2[X]$.

1. Show that f is an endomorphism of $\mathbb{R}_2[X]$.

2. Justify that the matrix A of f in $\mathcal{B}_0$ is written as follows :

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

3. Without additional calculations, determine using this matrix whether f is bijective or not.

4. Let the polynomials R_0, R_1, R_2 be defined by :

$$R_0 = 1,\ R_1 = X+1 \text{ and } R_2 = (X+1)^2$$

Show that $\mathcal{B}_1 = (R_0, R_1, R_2)$ is a basis of $\mathbb{R}_2[X]$.

5. (a) Determine the matrix D of f in the basis $\mathcal{B}_1$.

   (b) Briefly explain another method to determine this same matrix D. (No calculations are expected.)

6. Determine P, the transition matrix from the basis $\mathcal{B}_0$ to the basis $\mathcal{B}_1$, as well as Q, the transition matrix from the basis $\mathcal{B}_1$ to the basis $\mathcal{B}_0$.

7. (a) Give D^{-1}.

   (b) Write A^{-1} in terms of D^{-1}, P, and Q.

   (c) Let n be a natural number. Deduce from the previous question a simplified expression of $(A^{-1})^n$ in terms of D^{-1}, P, and Q.

8. Using some of the previous questions, answer the following :

   (a) Let $T \in \mathbb{R}_2[X]$ be fixed. How many solutions does the following differential equation (E_1) have in $\mathbb{R}_2[X]$:

$$(E_1) \qquad (x+1)y'(x) + y(x) = T(x) \quad ?$$

   (b) Determine, if there are any, all the solutions in $\mathbb{R}_2[X]$ of the following differential equation (E_2) :

$$(E_2) \qquad (x+1)y'(x) + y(x) = 6 + 6x \quad .$$

Exercise 5 ($\sim$ 6 points)

Part A : in $\mathbb{R}^3$

Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the endormorphism determined by the following matrix A in the canonical basis $\mathcal{B}$ of $\mathbb{R}^3$:

$$A = \begin{pmatrix} 4 & 0 & 0 \\ 2 & 3 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

1. Determine the set of all vectors $u \in \mathbb{R}^3$ satisfying $f(u) = 4u$.

2. Find a basis $\mathcal{B}'$ of $\mathbb{R}^3$ such that the matrix of f with respect to $\mathcal{B}'$ is $D = [f]_{\mathcal{B}'} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

3. What is the relationship between A, D and the transition matrix $[\mathcal{B}']_{\mathcal{B}}$?

4. Let Y be a matrix of $\mathcal{M}_3(\mathbb{R})$ such that $Y^2 = D$.
   (a) Show that $YD = DY$.
   (b) Deduce that Y must be a diagonal matrix.
   (c) Find all the possible Y satisfying $Y^2 = D$.

5. Using previous questions, explain how one can find all matrices $X \in \mathcal{M}_3(\mathbb{R})$ satisfying $X^2 = A$ (you are not asked to determine all such matrices).

Part B : on functions

Let $\varphi : \mathcal{F}(\mathbb{R}, \mathbb{R}) \to \mathcal{F}(\mathbb{R}, \mathbb{R})$ defined by $\varphi(f)(x) = f(x + \ln(4))$.

Let $G = \mathrm{Span}(\mathcal{B})$ with $\mathcal{B} = (\mathrm{ch}; \mathrm{sh})$.

6. Prove that $\mathcal{B}$ is a basis of G.

7. Show that $\varphi(\mathrm{ch}) \in G$ and $\varphi(\mathrm{sh}) \in G$ so we can consider the restricted endomorphism $\varphi : G \to G$.

8. Determine the matrix of φ with respect to the basis $\mathcal{B}$.

9. Let $f_1, f_2 \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ defined for all $x \in \mathbb{R}$ by $f_1(x) = e^x$ and $f_2(x) = e^{-x}$. Show that $\mathcal{B}' = (f_1; f_2)$ is a basis of G, and determine $P = [\mathcal{B}']_{\mathcal{B}}$.

10. Determine the matrix A' of φ with respect to $\mathcal{B}'$.

11. Find all matrices $X \in \mathcal{M}_2(\mathbb{R})$ satisfying $X^2 = A'$.

12. Take the only such X with only positive entries and show that it corresponds to the matrix (in $\mathcal{B}'$) of the linear application $\psi : G \to G$ defined by $\psi(f)(x) = f(x + \ln(2))$. Why could we expect such a result ?

---

0. Draw a ~~car~~ next to your name on the first page once this is done.
bicycle