End of Semester MNTES Exam

June the 14th 2024, 9h00. Duration: 2 hours

Not only your results, but above all, your ability to justify them clearly will be assessed. You are also reminded to take care with the spelling and presentation of your papers. All documents and calculators are forbidden. The grading scale is given as an indication only.

Exercice 1: Vector Fields (~ 4 pts)

Consider the following vector field expressions defined in $\mathbb{R}^2$, and the vector field figures in the Annex:

- $\vec{F_1} = 2y\vec{e_x} - x\vec{e_y}$
- $\vec{F_2} = 2y\vec{e_x} - \sin(x)\vec{e_y}$
- $\vec{F_3} = 2\frac{1}{x}\vec{e_x} - y\vec{e_y}$
- $\vec{F_4} = 2|x|\vec{e_x} - y\vec{e_y}$

1. Associate each vector field to one of the graphs in the Annex. A justification must be given for each choice. No point will be granted for simply guessing.
2. Calculate the fields lines for fields $\vec{F_1}$ and $\vec{F_3}$.
3. Which of these fields derive from a potential? Justify your answer.

Exercise 2: Field circulation and flux (~ 11 pts)

Consider the orthonormal Cartesian coordinate system $(O, \vec{e}_x, \vec{e}_y, \vec{e}_z)$ and the vector field $\vec{A}$ defined on $\mathbb{R}^3$ by

$$\vec{A}(x,y,z) = (y-z)\vec{e}_x + (x-z)\vec{e}_y + (\beta y - x)\vec{e}_z\,,$$

where β is a constant.

1. For what value(s) β_0 of the parameter β does the field $\vec{A}$ derive from a scalar potential?
2. In this question, we assume that β is *any value* (not necessarily equal to β_0).
   (a) Calculate the circulation of $\vec{A}$ along $\mathscr{C}_{OABO}$, boundary of the quarter disk $\mathscr{D}_{OAB}$, in the direction $OABO$.
   (b) Calculate the circulation of $\vec{A}$ along $\mathscr{C}_{OBCO}$, boundary of the quarter disk $\mathscr{D}_{OBC}$, in the $OBCO$ direction.
3. Still with the same consideration that β is *any value*, we'll now move on to the flux calculations.
   (a) Calculate the flux of $\vec{A}$ through the quarter disk $\mathscr{D}_{OAB}$, oriented along the outgoing normals to $\mathscr{B}$, using *cylindrical* coordinates (of axis Oz).

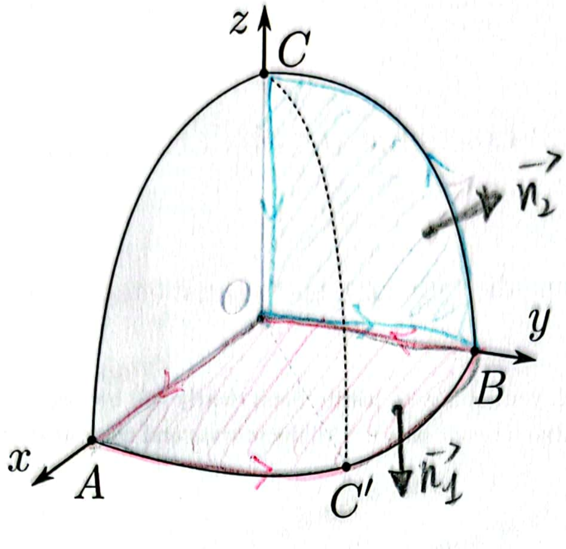


Figure 1: An eigth of a ball $\mathscr{B}$ is defined by : $\{0 \leqslant x, 0 \leqslant y, 0 \leqslant z, x^2+y^2+z^2 \leqslant R^2\}$

(b) Calculate the flux of $\vec{A}$ through the quarter-disk $\mathscr{D}_{OBC}$, oriented along the outgoing normals to $\mathscr{B}$, using *spherical* coordinates.

(c) Explain why the flux of $\vec{A}$ through $\mathscr{D}_{OAC}$ oriented by $+\vec{e}_y$ is equal to the flux of $\vec{A}$ through $\mathscr{D}_{OBC}$ calculated in the previous sub-question (full calculation of the first of these fluxes is not required).

(d) Using spherical coordinates, and directly applying its definition, give the expressions of the integrals to be calculated to find the flow of $\vec{A}$ through $\mathscr{S}_{ABC}$, oriented along the outgoing normal to $\mathscr{B}$. You will develop all the calculations, but without performing any of the integration calculations.

4. In this question, we assume that β is equal to the β_0 found previously.

   (a) Determine the scalar potential $V(x, y, z)$ from which $\vec{A}$ is derived (following the convention $\vec{A} = -\nabla V$) and verifying $V(0,0,0) = 0$.

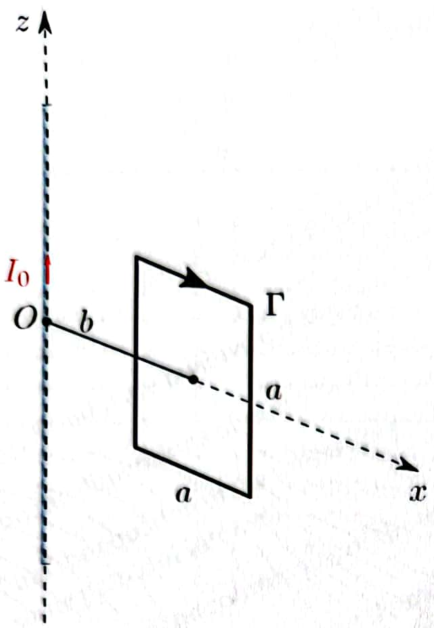
   (b) Determine the circulation of $\vec{A}$ along the quarter-circle joining points C and C', oriented from C to C', with C' the middle of arc AB. (cf. figure 1).

Exercice 3 : Fluxes in Physics ($\sim$ 3 pts)

Consider a cylindrical local frame $(M, \vec{e}_r, \vec{e}_\theta, \vec{e}_z)$ and cylindrical coordinates $M(r, \theta, z)$. A very long wire along the z-axis is trasversed by a constant current I_0 (expressed in Amperes [A]). The current I creates a magnetic field around the wire $\vec{B} = \frac{\mu_0 I_0}{2\pi r}\vec{e}_\theta$ (expressed in Teslas [T]) in the space surrounding the wire. A square loop, made from a conductor material, constitutes a path Γ of side a cm and is placed in the region close to the wire such that two of its sides are parallel to the wire. The center of the square loop is at a distance b from the wire. This setup is shown in figure 2(a), where the conventional direction of circulation around Γ is shown.

- Calculate the flux Φ_1 of $\vec{B}$ through the surface delimited by the square loop Γ. What happens to the flux Φ_1 if the square loop gets closer to the wire? And what if it gets further away?

- We now consider a 90° rotation of the square loop Γ around the x-axis. This new setup is shown in figure 2(b). Let Φ_2 be the flux of $\vec{B}$ through the surface delimited by the square loop Γ in this new configuration. Without any calculation, considering that $\Phi_2 = \lambda\Phi_1$ give the value of λ along with a justification.



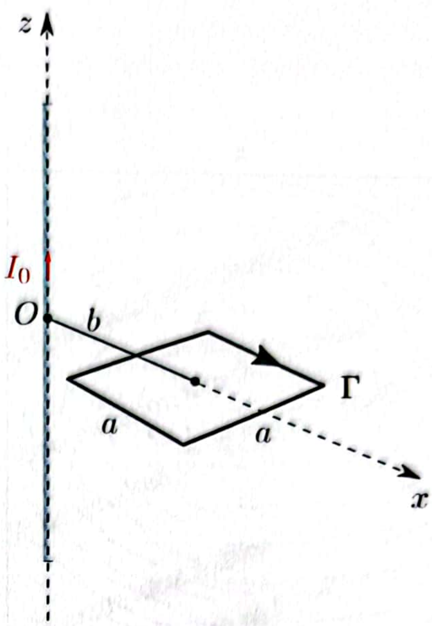


Figure 2: (a) Configuration A (b) Configuration B

Exercice 4: Field lines and Equipotentials (~ 2 pts)

Consider a local cylindrical frame $(M, \vec{e}_r, \vec{e}_\theta, \vec{e}_z)$ and a cylindrical coordinate system $M(r, \theta, z)$. Let a field $\vec{A} = r\vec{e}_r + z\vec{e}_z$.

1. Show that this field derives from a potential V. Determine the expressions of V and its family of equipotentials.

2. Determine the expression of the fields lines of $\vec{A}$. If you manage to do it without calculations, then give a proper justification for your answer.

Annex: Vector fields for Exercise 1