Semester 2 Final Physics Exam
June 18th 2024 – Duration: 3 h

Not only your results, but above all your ability to justify them clearly and analyze them critically will be accounted for. All results must be written as literal expressions involving only the data provided. You are also reminded to take care with the spelling and presentation.

Documents not allowed. Calculators in exam mode allowed. The grading scale is only indicative.

The 3 exercises are independent.

Exercise 1: Thrill rides (12 points)

This exercise is made of two independent parts (questions 1 to 6 independent from questions 7 to 10).

We consider the roller coaster depicted in figure 1. A spring of stiffness k and no-load length x_0, placed along the Ox direction, is used to propel a cart of mass m, **considered as a point mass**, that rolls on rails along a circuit containing a bump. The upper portion of the rail between A and C can be modeled as a circle of radius R_1.

The frame origin O is located at the fixed extremity of the spring.

In order to set the cart in motion, the spring is compressed until its length is x_i (start position), with $x_i < x_0$. To do so, an operator applies a force $\vec{F}$, opposed to the spring restoring force $\vec{F}_R$.

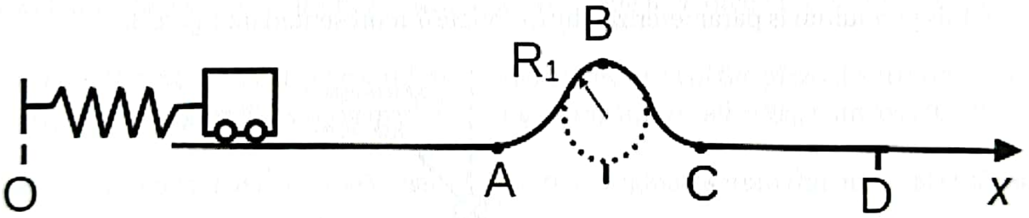

Figure 1: The Devil's Ride

For questions 1 and 2, section $[OA]$ is along axis (Ox) with $\vec{u}_x$ the unit vector.

The cart moves **without friction** all along its trajectory.

1. Draw a free-body diagram representing the forces applied on the cart at the start of the ride, **just after the spring was released**, with the spring still compressed in its starting position x_i. Compute the work W_O that the operator had to provide in order to compress the spring from its rest position x_0 to its start position x_i. Comment on the sign of this work.

The spring is suddenly released and propels the cart. The spring goes back to its length at rest x_0 **without additional spring extension**: the cart is released from the spring as soon as the spring reaches x_0.

2. Determine the cart velocity v_A at point A, as a function of k, m, x_0 and x_i.

The cart reaches now the "Crazy Bump" ABC. We recall that the upper part of the bump can be considered as a circle of radius R_1.

In order to describe the cart trajectory on the bump, we consider a new cylindrical frame $(O', \vec{u}_r, \vec{u}_\theta, \vec{u}_z)$, where O' is the center of the circle, $\theta = 0$ when the cart is on B, and $\vec{u}_z$ is the unit vector perpendicular to the paper (see figure 2 on the next page).

3. Determine the work W_{AI} of the cart's weight as it goes from point A to point I, parameterized by the angle θ.

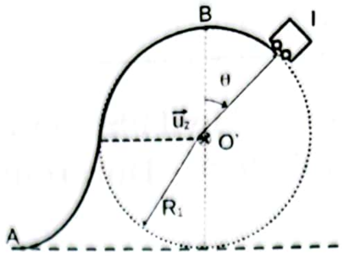

Figure 2: The Crazy Bump

4. Determine the magnitude of the normal reaction force of the rails $\vec{N}$ as the cart is on a point I located on the circular part of the bump, as a function of m, θ, v_A, g and R_1.

5. Under which condition will the cart take off and leave the rails?

6. Knowing that angle θ is in $[-\frac{\pi}{2};\frac{\pi}{2}]$, give the maximum velocity $v_{A,max}$ that the cart must not exceed in order to stay on the rails during the bump. Give the literal expression of the compression $(x_i - x_0)$ to which this corresponds, as well as the expression of the maximum force (F_{max}) that the operator should apply.

The following questions are independent of the previous ones.

After this not-so-scary "Devil's ride", our passengers try a second thrill ride, represented in figure 3. This one is a pendulum moving around an axis perpendicular to the plane of the scheme. It contains a counterweight located at a distance R_2 from the axis of rotation, of mass m_2 and moment of inertia $J_2 = m_2 R_2^2$. The cabin where the passengers are is located at a distance R_1 from the axis of rotation, with a mass m_1 and moment of inertia $J_1 = m_1 R_1^2$. The cabin is placed at an initial angle θ_0 by a motor, and then released without initial velocity. The whole system oscillates at its natural frequency ω_0.

The movement of this pendulum is parameterized by the angle θ represented on figure 3.

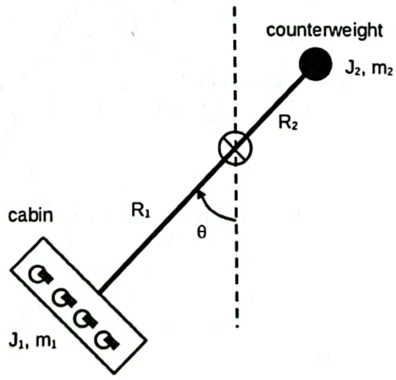

Figure 3: The Devil's Pendulum

7. **Considering that the angle θ remains small** in order for this ride not to be too scary, and that the pendulum oscillates **without friction**, establish the equation of motion of this pendulum and determine its natural angular frequency ω_0. The weight of the bar between the two masses is neglected and the equation should involve only m_1, m_2, R_1, R_2, g and θ. Comment on the case where $m_2 R_2 > m_1 R_1$.

8. Assuming $m_2 R_2 < m_1 R_1$, give the expression of $\theta(t)$.
For simplicity, we consider now that $R_1 = R_2 = R$.

9. Noting ω_{00} the pendulum natural angular frequency **without counterweight**, determine the ratio $\frac{\omega_{00}}{\omega_0}$. What is the interest of the counterweight?.

10. Numerical application: determine the ratio $\frac{m_2}{m_1}$ for $R = 5$ m so that $\omega_0 = 1$ rad.s^{-1}, with $g = 9.81$ m.s^{-2}.

11. *Bonus:* if the total mass of the passengers is 750 kg, how can one ensure that the period of oscillations does not depend on the number of passengers?

Exercise 2: Quartz, an excellent resonator (12 points)

Quartz is a piezoelectric crystalline mineral: it deforms when submitted to a voltage difference, and conversely if it is deformed mechanically a voltage difference appears between its faces. A quartz crystal undergoes mechanical vibrations at a very specific frequency. The accuracy of the resonant frequency, associated to electric coupling thanks to the piezoelectric effect, makes quartz a useful component for designing resonant electric circuits with a high quality factor, and thus very accurate oscillators.

Part 1: modeling a quartz resonator

We consider a quartz crystal shaped as a thin disk. The circular base has a diameter $d = 1$ cm with a crystal thickness e equal to 0.2 mm. Metal electrodes (usually made in gold) are placed on each circular face of the quartz; they are referred to as connection electrodes. This results in a plate capacitor structure as show in the figure below.

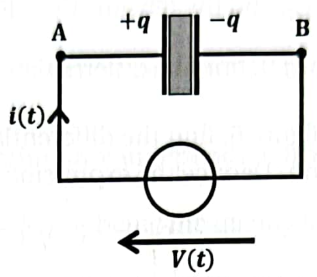

Figure 4: Electric scheme of a quartz under a voltage difference $V(t)$.

From a mechanical point of view, when the piezoelectric system is submitted to an alternating sinusoidal voltage difference $V(t)$, it will undergo mechanical sinusoidal oscillations, due to a force proportional to the voltage difference.

We set $V(t) = V_0\cos(\omega t)$. The model is the following: an element of mass m of the piezoelectric body, placed at a distance x from its resting position (see figure 5), is subjected to the following forces, all along with axis (Ox):

- a restoring elastic-type force $\vec{F} = -kx\vec{u}_x$ (with $k > 0$), that originates from the material stiffness,
- friction effects resulting in a force assumed to be proportional to the velocity $\vec{f} = -h\frac{dx}{dt}\vec{u}_x$,
- a force resulting from the piezoelectric effect $\vec{F}_{PE} = \beta V(t)\vec{u}_x$ (with $\beta > 0$),
- the weight of the element, considered here to be negligible.

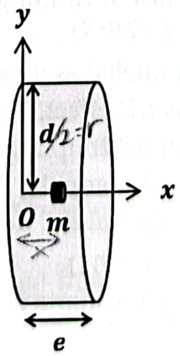

Figure 5: Small element of mass m located at x

1. The motion is assumed to be restricted to the (Ox) axis. Apply Newton's second law to the element of mass m in the laboratory frame (supposed to be Galilean) to determine the differential equation satisfied by $x(t)$. Which term corresponds to the electromechanical coupling?

From an electrical point of view, we denote $q(t)$ the total charge carried by the plate capacitor (see Fig. 4). This charge originates from 2 contributions:

- a regular capacitance charge $q_1(t)$ issued from the metal/crystal/metal capacitor. In the following we will denote C_p the corresponding capacitance and refer to it as the *connection capacitance.* It is admitted that the capacitance writes $C_p = \frac{\epsilon_0 \epsilon_r S}{e}$, S being the electrode area, e the thickness of the crystal disk, ϵ_0 the vacuum permittivity and ϵ_r a dimensionless constant depending on the piezoelectric material (see figure 6)
- a charge induced by the piezoelectric effect $q_2(t)$ which is proportional to $x(t)$: $q_2(t) = \gamma x(t)$. This effect is equivalent to adding **in parallel to** C_P a circuit composed of a resistance R, a coil of inductance L and a capacitor of capacitance C_S (see figure 6).

We have therefore $q(t) = q_1(t) + q_2(t)$.
We give: $\epsilon_0 = 8.85 \times 10^{-12}$ F·m^{-1} ; $\epsilon_r = 2.3$

2. Recall the link between $q_1(t)$, capacitance C_p and the tension $V(t)$. Find the numerical value of C_p.
3. Using the differential equation satisfied by $x(t)$, find the differential equation satisfied by $q_2(t)$.
4. Using the equivalent circuit presented in figure 6, find the differential equation satisfied by $q_2(t)$ and compare it to the one obtained in the previous question. Deduce the expression of R, L, C_s in terms of m, h, β, γ and k.

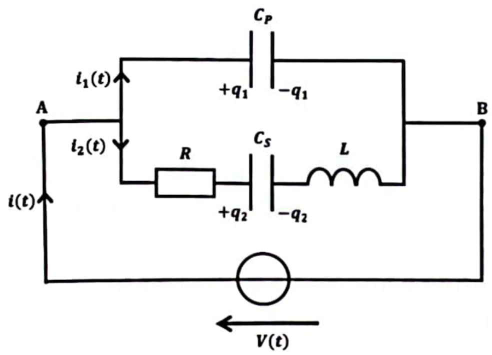

Figure 6: Equivalent electrical circuit of the quartz oscillator

Part 2: Experimental study of the resonance of a quartz oscillator

In order to study the sharp resonance of the quartz oscillator, we subject it to AC voltages at different frequencies. We then measure the amplitude of the current intensity running through the quartz at the various frequencies.The following study will be conducted in the forced permanent AC regime using complex notations. We have: $\underline{i}(t) = \frac{\underline{V}(t)}{\underline{Z}_q}$, $\underline{Z}_q$ being the impedance of the quartz. I is the amplitude of the current running through the quartz and ϕ the phase-shift between $V(t)$ and $i(t)$ such that $i(t) = I\cos(\omega t + \phi)$ and $\underline{i}(t) = Ie^{j(\omega t+\phi)}$.

As seen in the previous part, the quartz can be modeled as a capacitor C_p (the connection capacitance) in parallel with a series R, L, C_S circuit representing the piezoelectric effect. The latter circuit (*i.e.* parallel R, L, C_S) is often referred as the *motional* circuit and models the electromechanical coupling induced by the piezoelectric effect.

In order to obtain resonance phenomena, we look for angular frequencies ω such that the amplitude of $\underline{i}(t)$ (*i.e.* I) reaches large values, that is to say that $\frac{1}{\underline{Z}_q}$ tends to large values. To find the resonance, we first neglect any dissipative effect : **in the following two questions we will assume** $R = 0\,\Omega$.

The figure below then shows a simplified model of the quartz resonator.

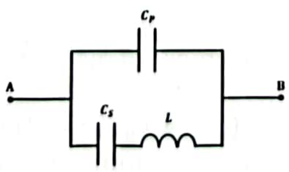

Figure 7: Simplified electrical model of the quartz resonator

5. Show that $\underline{Z}_q$, the equivalent impedance of dipole AB verifies:

$$\frac{1}{\underline{Z}_q} = jC\omega \times \frac{1-\frac{\omega^2}{\omega_2^2}}{1-\frac{\omega^2}{\omega_1^2}}$$, with: $\omega_1 = \frac{1}{\sqrt{LC_S}}$. ω_2 and C are constants to be determined in terms of C_p, C_S and L.

6. Deduce the expression of the frequency for which we should observe resonance of the current intensity. We will denote it f_1.

Previous questions have shown that the R, L, C_S branch is responsible for the resonance phenomena. In the following we will then neglect the capacitance C_P, focusing ourselves on the circuit depicted in figure 8.

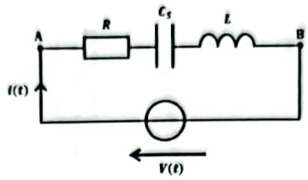

Figure 8: Simplified quart model to study resonance phenomenon

The following questions are independent from the previous ones.

7. Show that: $\underline{i}(t) = \frac{\frac{V(t)}{R}}{1+jQ\left(\frac{\omega}{\omega_1}-\frac{\omega_1}{\omega}\right)}$, with: $Q = \frac{1}{R}\sqrt{\frac{L}{C_S}}$ being the quality factor and: $\omega_1 = \frac{1}{\sqrt{LC_S}}$.

The curve of figure 9 shows the evolution of the current intensity amplitude I of $\underline{i}(t)$ against the frequency f of the input voltage $V(t)$. The amplitude of $V(t)$ is denoted V_0 and is worth (0.20 ± 0.02) V.

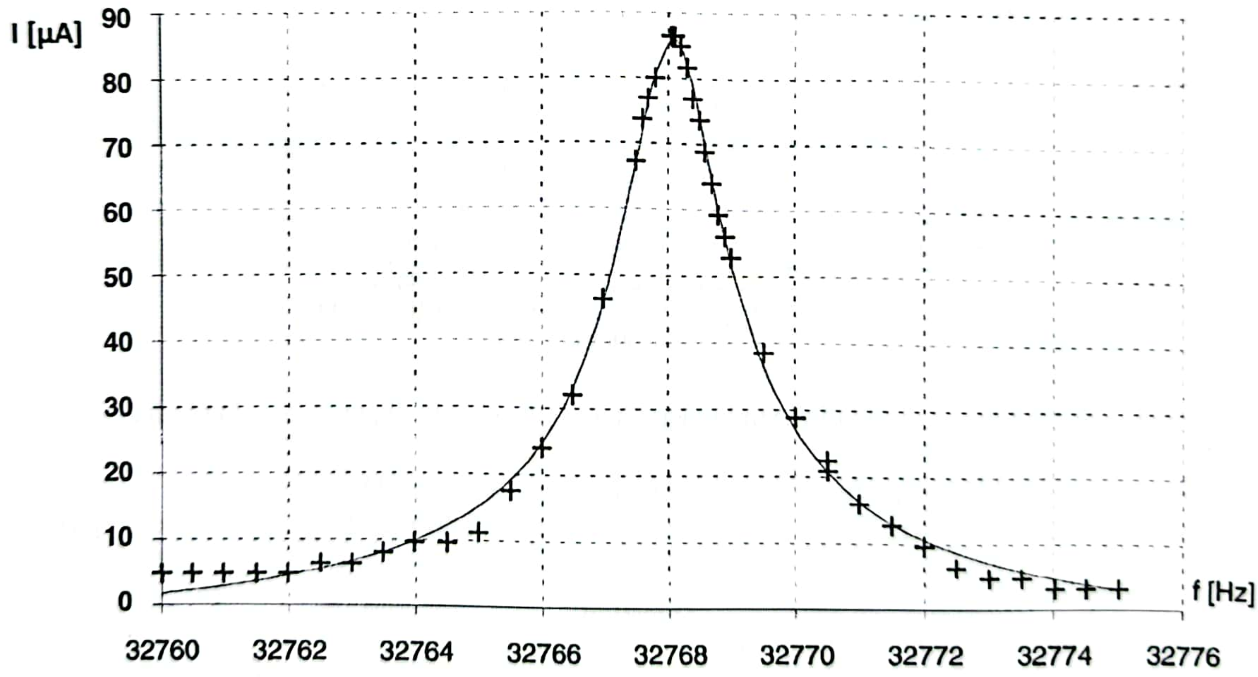

Figure 9: Source: Dieber, Kempf et Markiewicz, BUP n°799, 1997

We also remind that the quality factor Q is defined from: $Q = \frac{f_r}{\Delta f}$, f_r being the resonance frequency and Δf the bandwidth. We have $\Delta f = |f_{C_2} - f_{C_1}|$, f_{C_1} and f_{C_2} being the two cutoff frequencies.

8. Using the graph of Figure 9, find the value of R together with its uncertainty.

9. Determine the value of Q (without uncertainty). Comment on the value.

In the following we consider the following approximate values: $R \simeq 2$ kΩ, $Q \simeq 2 \times 10^4$ and: $\omega_1 \simeq 2 \times 10^5$ rad/s.

10. Give the expression of L and C_S in terms of Q, R and ω_1.

11. Deduce the value of L (without uncertainty). Comment on the value.

Exercise 3: Hidden dipoles (6 points)

This exercise is an open problem, it is therefore important that one can follow and understand your reasoning. Your ability to highlight a question, write an organized reasoning (even without reaching a conclusion) and have a critical look on your results will be assessed.

With a single resistor, a coil and a capacitor, we form two dipoles D_1 and D_2. This dipoles are used to form a second order filter as shown on the figure below.

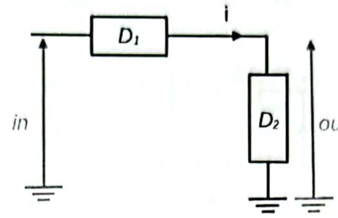

Figure 10: Scheme of the unknown filter

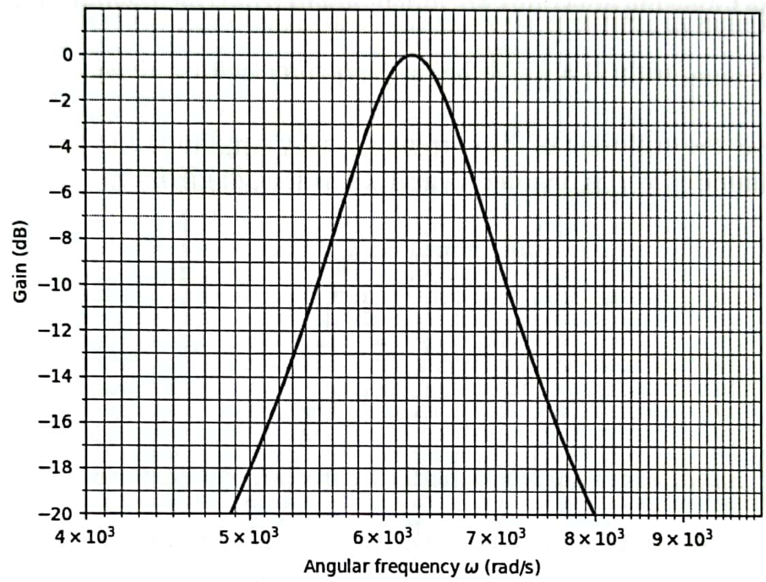

Figure 11: Bode diagram of the unknown filter for the gain in dB, on a semi-log scale.

In DC regime, we measure a current intensity of 1 mA for an input voltage of 3 V.
In AC regime, the filter behaves as a band-pass filter, and the Bode diagram for the gain in decibels is given above.

Question

Identify the dipoles D_1 and D_2 by studying the asymptotic behavior at very low and very high frequencies, and determine the values of the resistance, the inductance and the capacitance of the three components.

Data

The transfer function of a band-pass second order filter is:

$$\underline{H} = \frac{H_0}{1+jQ\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)} = \frac{H_0\frac{j\frac{\omega}{\omega_0}}{Q}}{1-(\frac{\omega}{\omega_0})^2+\frac{j\frac{\omega}{\omega_0}}{Q}}$$

with H_0 the gain coefficient, ω_0 the characteristic angular frequency and Q quality factor, that can also be defined using the characteristic and cut-off angular frequencies: $Q = \frac{\omega_0}{|\omega_{c1}-\omega_{c2}|}$.