Physics: Test n°1

Monday November 18, 2024 | Duration : 1h30

Unless otherwise stated, all answers must be justified. The marking scale is only indicative. The two exercises are independent. **A one-A4 page handwritten form (recto) is authorized.**

Calculator in exam mode authorized.

A formula sheet is provided at the end of the subject.

Exercise 1: Modeling the charge of a proton (~10 pts)

There are several highly simplified classical models of the electric field created by the proton throughout space. Here, we present one of the simplest models.

We'll consider the proton as a charged particle, **its charge having a spherical symmetry.** In the model below, almost all of the proton's charge (over 99.9%) is concentrated in an extremely small volume (a sphere with a radius of the order of 10^{-15} m), but **its charge extends to infinity**, asymptotically tending towards 0 at infinity.

The electric field created by the proton throughout space ($0 < r < \infty$) is, in a vacuum of permittivity ε_0, of the form (in spherical coordinates, see figure 3 of the formula sheet):

$$\vec{E} = \frac{A}{r^2}\exp\left(-\frac{b}{r}\right)\vec{u}_r$$

$$\text{where: } A = \frac{e}{4\pi\varepsilon_0}$$

e is the proton charge and b is a characteristic parameter of the spatial charge distribution, expressed as a function of various universal constants. We give $b = 0.161\ 10^{-15}$ m.

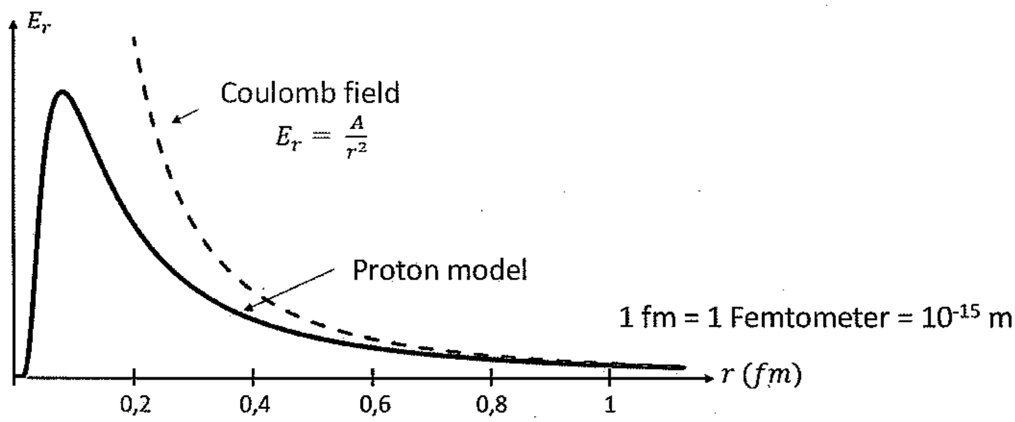

Figure 1: Electric field created by the proton (E_r = radial component)

1.1 Comparison with the Coulomb model

Q1: Figure 1 compares this electric field $\vec{E}$ with the Coulomb electric field $\vec{E}_c = \frac{A}{r^2}\vec{u}_r$.

a) Recall the value of the elementary charge e, with two significant figures.

b) Compare and comment on these 2 curves. What are the advantages of one model over the other? In particular, can you justify, by studying the symmetries of the charge distribution, the zero value of the proton's electric field at r = 0?

c) What must be the ratio $\frac{b}{r}$ so that the Coulomb field $\vec{E}_c$ approaches this model to less than 1%?
Numerical application: specify at what distance from the centre the electric field value given by the Coulomb model is identical to within 1% (it will actually be greater than that of the proton model) to that given by the proton model.

1.2 Proton volume charge density

In the following questions, only the <u>proton model</u> will be used, and it will be assumed that Maxwell's equations are also valid at the scale of elementary particles.

Q2: Determine the proton's volume charge density ρ(r), as a function of A, b, ε_0 and **r**.

1.3 Proton charge

Q3: Write the integral, without calculating it but specifying the integration limits, to determine Q, the proton's charge.

- **Q4:** Determine this charge and show that Q = e.
The following primitive function is given:
$\int \frac{1}{r^2} \exp(-\frac{b}{r})\mathrm{d}r = \frac{1}{b} \exp(-\frac{b}{r}) + K,\ K \in \mathbb{R}$

1.4 Electric potential

Q5: Determine the expression V(r) for the electric potential associated with the proton, assuming it is nil at infinite distance from the proton.

Q6: Simplify this relationship for r > 8b, knowing that in this case we have, to within 1%:
$\exp\left(-\frac{b}{r}\right) \approx 1 - \frac{b}{r}$

1.5 Electrostatic energy

Q7: Using the volume density of electrostatic energy, write the literal expression for the electrostatic energy W_e of the proton under this model, first as a function of A, ε_0 and b, then as a function of e, ε_0 and b.

Exercise 2: Dihedral capacitor (~10 pts)

We're interested in a dihedral capacitor, made up of two identical, non-parallel metal plates: the angle between these two plates is called α (see figure 2). The plates have a <u>non-negligible thickness,</u> a width h in the direction parallel to (Oz) and a length equal to b-a. The medium between the plates is air of permittivity ε_0. The capacitor is powered by a generator imposing a constant potential difference U_0 between the two plates, as shown in figure 2. Edge effects are neglected.

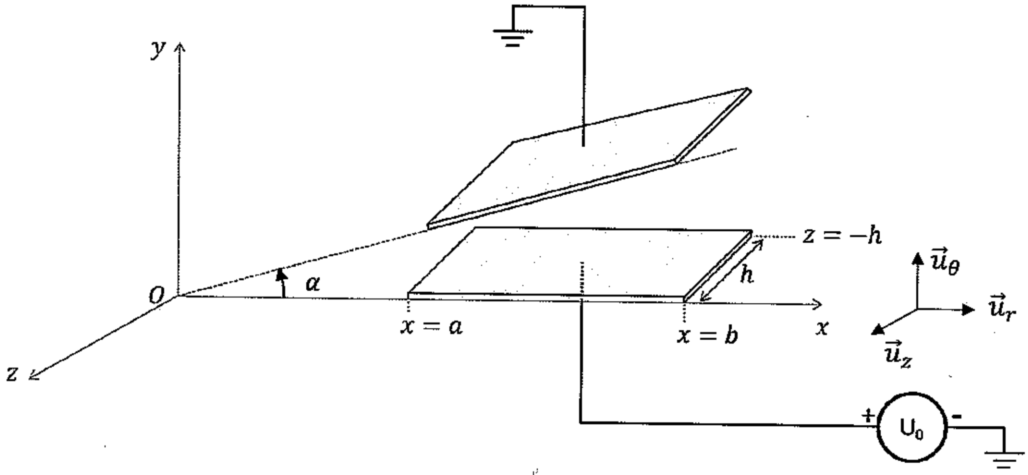

Figure 2: Diagram of the dihedral capacitor

It is assumed that the electrostatic potential, between the two plates, depends only on the variable θ of the cylindrical reference frame with axis (Oz): V = V(θ).

Q8: Draw, on a diagram, equipotential surfaces between the two plates of this capacitor.
Without prior calculation:

- Draw the electric field lines $\vec{E}$ between the plates and justify them.
- Where is the norm of $\vec{E}$ maximum?

Q9: Starting from one of Maxwell equations, show how to get to Poisson's equation:

$$\Delta V = -\frac{\rho}{\varepsilon_0}$$

Recall that the Laplacian of a scalar field corresponds to the divergence of the gradient of that field.

X **Q10:** By solving the Poisson's equation between the plates, $\Delta V = -\frac{\rho}{\varepsilon_0}$ and by using boundary conditions shown in Figure 2, show that: $V(\theta) = U_0(1 - \frac{\theta}{\alpha})$

Q11: Deduce the literal expression of $\vec{E}$ between the capacitor plates, as a function of U_0, r and α. Is this expression consistent with your answers to Q8?

Q12: Make a diagram of the interface at θ = 0, i.e. between the lower capacitor plate and the air in the inter-plate space. Use this diagram and a boundary condition applying to the electric field $\vec{E}$ to show that the surface charge density on the positively charged plate is given by:

$$\sigma(r) = \frac{U_0\,\varepsilon_0}{\alpha\,r}$$

Q13: Determine the literal expression of the total charge, Q, carried by the positively charged plate, as a function of U_0, ε_0, h, α, b and a.

Q14: From the previous question, deduce the literal expression for the capacitance, C, of this capacitor.

Formula sheet

Operators in cylindrical coordinates

$$\vec{\nabla} U = \vec{grad}\, U = \frac{\partial U}{\partial r}\vec{u}_r + \frac{1}{r}\frac{\partial U}{\partial \theta}\vec{u}_\theta + \frac{\partial U}{\partial z}\vec{u}_z$$

$$\vec{\nabla} \cdot \vec{E} = div\, \vec{E} = \frac{1}{r}\frac{\partial}{\partial r}(r\, E_r) + \frac{1}{r}\frac{\partial E_\theta}{\partial \theta} + \frac{\partial E_z}{\partial z}$$

$$\vec{\nabla} \wedge \vec{E} = \vec{rot}\, \vec{E} = \left(\frac{1}{r}\frac{\partial E_z}{\partial \theta} - \frac{\partial E_\theta}{\partial z}\right)\vec{u_r} + \left(\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r}\right)\vec{u_\theta} + \left(\frac{1}{r}\frac{\partial}{\partial r}(r\, E_\theta) - \frac{1}{r}\frac{\partial E_r}{\partial \theta}\right)\vec{u_z}$$

$$\nabla^2 V = \Delta V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2}$$

Convention of the spherical system used in this subject

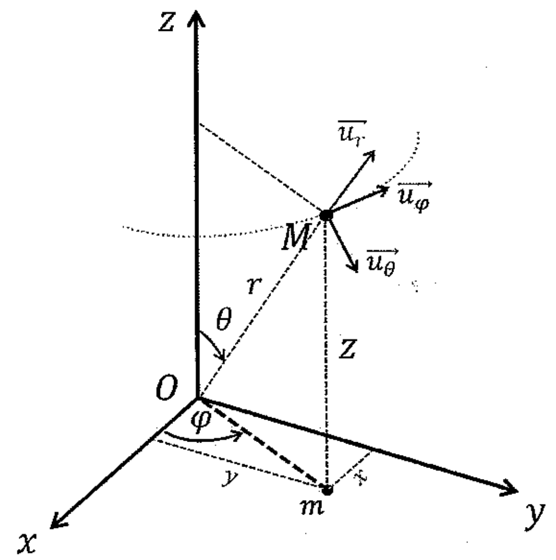

Figure 3: Spherical coordinates and associated local coordinate system

Operators in spherical coordinates

$$\vec{\nabla} U = \vec{grad}\, U = \frac{\partial U}{\partial r}\vec{u}_r + \frac{1}{r}\frac{\partial U}{\partial \theta}\vec{u}_\theta + \frac{1}{r sin\,\theta}\frac{\partial U}{\partial \phi}\vec{u}_\phi$$

$$\vec{\nabla} \cdot \vec{E} = div\, \vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\, E_r) + \frac{1}{r\, sin\,\theta}\frac{\partial}{\partial \theta}(sin\,\theta\, E_\theta) + \frac{1}{r\, sin\,\theta}\frac{\partial E_\phi}{\partial \phi}$$

$$\vec{\nabla} \wedge \vec{E} = \vec{rot}\vec{E} = \left\{\frac{1}{r sin\theta}\left[\frac{\partial}{\partial \theta}(E_\phi sin\theta) - \frac{\partial E_\theta}{\partial \phi}\right]\right\}\vec{u_r}$$
$$+ \left\{\frac{1}{r}\left[\frac{1}{sin\theta}\frac{\partial E_r}{\partial \phi} - \frac{\partial}{\partial r}(r E_\phi)\right]\right\}\vec{u_\theta}$$
$$+ \left\{\frac{1}{r}\left[\frac{\partial}{\partial r}(r E_\theta) - \frac{\partial E_r}{\partial \theta}\right]\right\}\vec{u_\phi}$$

$$\nabla^2 V = \Delta V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial \theta}\left(sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$