INSA LYON

FIMI - SCAN 2

MATH TEST # 2 ON DECEMBER 16, 2024 - DURATION 1H30

Warnings and Advice

- All documents, dictionnaries, calculators or electronic devices, and communication means are prohibited.
- The marking scheme is provided for reference only.
- Presentation, quality of writing, clarity, and precision of reasoning are taken into account in the grading.

EXERCICE 1 *(4 pts)*

The questions 1, 2, and 3 are independent.

1) Let $A = \begin{pmatrix} 2 & 0 & 0 \\ 15 & -1 & 0 \\ -17 & 13 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -11 & 65 \\ 0 & 1 & -7 \\ 0 & 0 & 1 \end{pmatrix}$, and $C = \begin{pmatrix} 2 & -22 & 130 \\ 15 & -166 & 982 \\ -17 & 200 & -1193 \end{pmatrix}$.

   It is given that $C = AB$. Determine $\det(C)$.

2) a) Let $a \in \mathbb{R}$, and $M_a = \begin{pmatrix} a-2 & 2 & -1 \\ 2 & a & 2 \\ 2a & 2a+2 & a+1 \end{pmatrix}$. Compute $\det(M_a)$ in terms of a.

   b) For which values of $a \in \mathbb{R}$ is the matrix M_a invertible?

3) Let f be the endomorphism of $\mathbb{R}^3$ whose matrix in the standard basis is :

$$A = \begin{pmatrix} 3 & 1 & -1 \\ -2 & -1 & 2 \\ -2 & -3 & 4 \end{pmatrix}$$

   a) Let $u = (1, 0, 1)$. Verify that u is an eigenvector of f. To which eigenvalue λ is it associated?

   b) It is given that $\mu = 1$ is also an eigenvalue. Without computing the determinant, determine the third eigenvalue and provide the characteristic polynomial of A.

EXERCICE 2 *(4.5 pts)*

We aim to study the sequence $(u_n)_{n\in\mathbb{N}}$ defined recursively by $u_0 = 0$ and for all $n \in \mathbb{N}$, $u_{n+1} = f(u_n)$, where $f : x \mapsto (1+x)\mathrm{e}^{-x}$.

1) Study the variations of f on $[0, +\infty[$ and verify that the interval $[0, 1]$ is stable under f.
2) By studying the function $h : x \mapsto f(x) - x$, show that f has a unique fixed point α on $[0, +\infty[$ and that $\alpha < 1$ (one will not try to determine α).
3) Show that $\dfrac{2}{\mathrm{e}} < \alpha < 1$.
4) It is given that $f \circ f$ has a unique fixed point on $[0, +\infty[$. Show that this fixed point is equal to α.
5) Using the sequences $(u_{2n})_{n\in\mathbb{N}}$ and $(u_{2n+1})_{n\in\mathbb{N}}$, show that (u_n) converges to α.

EXERCICE 3 *(11.5 pts)*

Part I

Let $(x_n)_{n\in\mathbb{N}}$ be a real sequence. We assume that there exist $\alpha \in \mathbb{R}_+^*$ and $n_0 \in \mathbb{N}$, such that $(x_n)_{n\in\mathbb{N}}$ satisfies the following property (P) :

$$(P): \quad \forall n \geqslant n_0, \quad x_{n+1} - x_n \geqslant \frac{\alpha}{n}.$$

1) Let $n \in \mathbb{N}^*$. Show that

$$x_{2n} - x_n = \sum_{k=n}^{2n-1} (x_{k+1} - x_k).$$

2) Deduce that

$$\forall n \geqslant n_0,\ x_{2n} - x_n \geqslant \frac{\alpha}{2}.$$

3) Show properly that $\lim\limits_{n\to+\infty} x_n = +\infty$.

Part II

Let f be defined on $[0\,,1]$, by $f(x) = x - x^2$. Let $(u_n)_{n\in\mathbb{N}}$ be the sequence defined by :

$$\forall n \in \mathbb{N}, u_{n+1} = f(u_n) \quad ; \ u_0 = \frac{1}{2}.$$

1) Study the variations of f (on $[0\,,1]$).
2) Show that

$$\forall n \in \mathbb{N},\ f\left(\frac{1}{n+1}\right) \leqslant \frac{1}{n+2}.$$

3) Deduce that

$$\forall n \in \mathbb{N},\ 0 < u_n < \frac{1}{n+1}.$$

4) For all $n \in \mathbb{N}$, define $v_n = nu_n$. Show that $(v_n)_{n\in\mathbb{N}}$ is increasing.
5) Deduce that $(v_n)_{n\in\mathbb{N}}$ converges and that its limit ℓ belongs to $[0\,,1]$.
6) Justify that $\ell \neq 0$.
7) For all $n \in \mathbb{N}$, define $w_n = n(v_{n+1} - v_n)$. Express w_n in terms of n and u_n. Deduce that $(w_n)_{n\in\mathbb{N}}$ converges to $\ell(1-\ell)$.
8) In this question, we will prove by contradiction that $\ell = 1$. We thus assume that $\ell \neq 1$. Justify that there exist $\alpha \in \mathbb{R}$ and $n_0 \in \mathbb{N}$, such that for all $n \geqslant n_0$, $v_{n+1} - v_n \geqslant \dfrac{\alpha}{n}$. Conclude using Part I and determine an asymptotic equivalent of u_n.