INSA LYON

FIMI – SCAN 2

MATH TEST #1 ON NOVEMBER 4, 2024 - DURATION 1H30

Warnings and Advice

- All documents, calculators or electronic devices, means of communication, are prohibited.
- The grading scale is provided as a guide.
- Presentation, quality of writing, clarity, and precision of reasoning are taken into account in grading.

EXERCISE 1 *(7.5 points)*

The questions in this exercise are independent of each other.

1) Determine the nature of the following integrals :

a) $\displaystyle\int_0^{+\infty} \mathrm{e}^{-\sqrt{t}}\left(1+3t^4\right)\,\mathrm{d}t$ b) $\displaystyle\int_1^{\mathrm{e}} \frac{t+2}{\ln(t)}\,\mathrm{d}t.$

2) Show the convergence and compute $\displaystyle\int_4^{+\infty} \frac{1}{x^2-9}\,\mathrm{d}x.$

You may write $\dfrac{1}{x^2-9}$ in the form $\dfrac{a}{x-3}+\dfrac{b}{x+3}$, where a and b are real numbers to be determined.

3) Let $\alpha \in \mathbb{R}$ and $I_\alpha = \displaystyle\int_0^{+\infty} \frac{1}{x^\alpha}\arctan\left(\frac{1}{x^3}\right)\mathrm{d}x.$

Determine the values of α for which the integral I_α is convergent.

4) Define the sequences $(u_n)_{n\in\mathbb{N}^*}$ and $(v_n)_{n\in\mathbb{N}^*}$ by

$$\forall n \in \mathbb{N}^*, \quad u_n = \sum_{k=0}^{n} \frac{1}{k!} \text{ and } v_n = u_n + \frac{1}{n!}.$$

Show that $(u_n)_{n\in\mathbb{N}^*}$ and $(v_n)_{n\in\mathbb{N}^*}$ are adjacent.

What can be deduced from this ?

Exercise 2 *(6.5 points)*

1) Show that the integral $I = \displaystyle\int_0^{+\infty} \frac{\sin^3(t)}{t^2}\,dt$ converges.

The goal of the following questions is to determine its value.

2) Let $x > 0$. Define $J(x) = \displaystyle\int_x^{+\infty} \frac{\sin^3(t)}{t^2}\,dt$, which converges according to the previous question.

- a) Show that $\displaystyle\int_x^{+\infty} \frac{\sin(3t)}{t^2}\,\mathrm{d}t = 3\int_{3x}^{+\infty} \frac{\sin(u)}{u^2}\,\mathrm{d}u.$

  (We admit the convergence of these integrals).

- b) Assuming that for any real number t, $\sin^3(t) = \dfrac{3\sin(t) - \sin(3t)}{4}$, show that

$$J(x) = \frac{3}{4}\int_x^{3x} \frac{\sin(t)}{t^2}\,dt.$$

3) Let φ be the function defined on $]0, +\infty[$ by : $\forall t > 0,\ \varphi(t) = \dfrac{\sin(t) - t}{t^2}$.

a) Show that φ has a finite limit at 0.

Thus, it can be extended by continuity at 0, and we will still denote by φ the function thus extended, continuous on $[0, +\infty[$.

b) Show that

$$\forall x > 0,\ \int_x^{3x} \frac{\sin(t)}{t^2}\,\mathrm{d}t = \int_x^{3x} \varphi(t)\,\mathrm{d}t + \ln(3).$$

4) Deduce the value of I from the previous questions.

Exercise 3 *(6 points)*

For all $n \in \mathbb{N}^*$, let $I_n = \displaystyle\int_0^{+\infty} \frac{1}{(1+t^3)^n}\,\mathrm{d}t.$

1) Show that the integral I_n converges for every $n \in \mathbb{N}^*$.

2) Determine the monotonicity of the sequence $(I_n)_{n\in\mathbb{N}^*}$ and deduce that it converges to a limit $\ell \geqslant 0$.

3) Using integration by parts, show that

$$\forall n \in \mathbb{N}^*,\ I_n = 3n\left(I_n - I_{n+1}\right).$$

4) Determine $\displaystyle\lim_{n\to+\infty}\left(1 + \frac{1}{\sqrt{n}}\right)^n$.

5) For any $n \in \mathbb{N}^*$, prove the three inequalities below :

$$\int_0^{\frac{1}{\sqrt[6]{n}}} \frac{1}{(1+t^3)^n}\,\mathrm{d}t \leqslant \frac{1}{\sqrt[6]{n}}$$

$$\int_{\frac{1}{\sqrt[6]{n}}}^{1} \frac{1}{(1+t^3)^n}\,\mathrm{d}t \leqslant \frac{1 - \frac{1}{\sqrt[6]{n}}}{\left(1 + \frac{1}{\sqrt{n}}\right)^n}$$

$$\int_1^{+\infty} \frac{1}{(1+t^3)^n}\,\mathrm{d}t \leqslant \frac{1}{3n-1}.$$

6) Deduce the value of ℓ.
