1^{st} MNTES exam – Semester 2
April 5^{th}, 2024. Duration: 1.5 h

No document allowed. No mobile phone, no electronic devices (like apple watch). Non-programmable calculator allowed. The proposed grading scale is only indicative.
It is also reminded that the general clarity and cleanness of your paper may also be taken into account.

Exercise 1 (4 points)

Sketch the domain Δ then calculate the integral.

$$\iint_\Delta f(x,y)\ \mathrm{d}x\mathrm{d}y \quad \text{with} \quad f(x,y)=\frac{1}{1+x^2+y^2},$$

and $\Delta = \{(x,y)\in\mathbb{R} \mid 0\le x\le 1,\ 0\le y\le 1,\ x^2+y^2\le 1\}$. The integral must be first expressed using Cartesian coordinates, then perform a change of variables. (Hint rewrite Δ as normal in x or y to help pose the double integral)

Exercise 2 Yukawa model of the hydrogen atom (7 points)

The diagram of an element of elementary volume ($\mathrm{d}V$) in spherical coordinates is given in Figure 2 (see Annex).

1. Complete the diagram in the Annex, explicitly indicating all the terms necessary to derive the expression for $\mathrm{d}V$.

2. Show that, for a function $f(\rho)$ depending only on the variable ρ, the integral of $f(\rho)$ over a ball B of radius R, in spherical coordinates, can be expressed as a simple integral

$$\iiint_B f(\rho)\ \mathrm{d}V = \int_0^R f(\rho)\cdot 4\pi\rho^2\ \mathrm{d}\rho.$$

3. Application: In Yukawa's hydrogen atom model the volume density of the electron charge is given by

$$\Psi(\rho) = -q\cdot\frac{\mathrm{e}^{-\frac{\rho}{a}}}{4\pi a^2\rho},$$

with ρ the distance to the origin, a a constant having the dimension of a length, and q the constant elementary charge.

   (a) Determine the total charge Q contained in a sphere of radius R. For this we will have to use integration by parts to obtain this result. Recall that

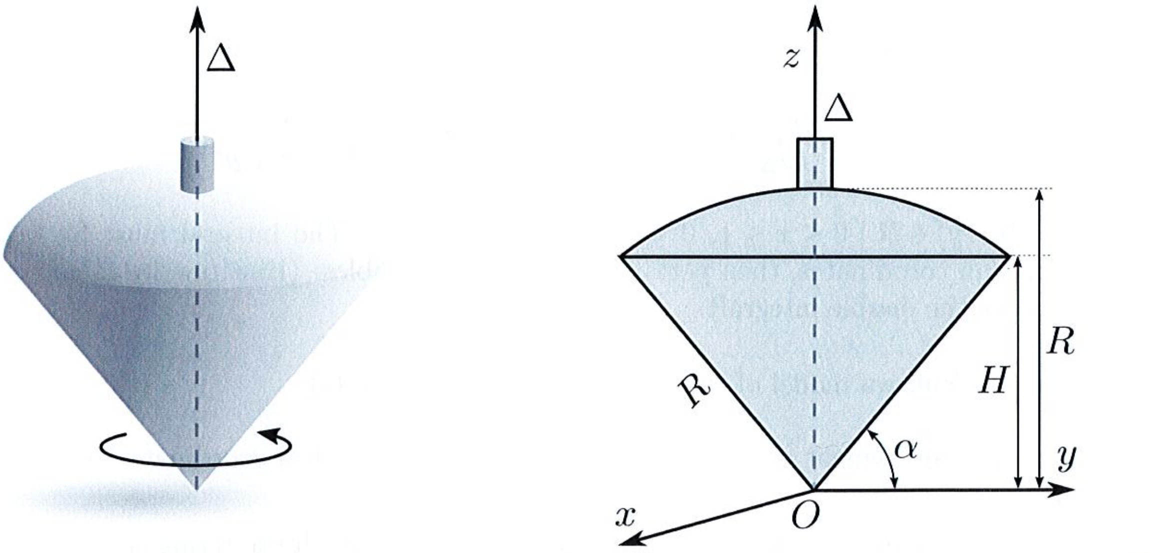
$$\int_a^b u(x)v'(x)\ \mathrm{d}x = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x)\ \mathrm{d}x$$

   (b) What is the value of this integral when R tends to infinity? Justify your answer.

   (c) According to this model, where is the barycenter of the hydrogen atom's charges?

Exercise 3 Geometry of masses (6.5 points)

Consider a spinner of homogeneous mass density μ, whose shape is shown in Figure 1 (it consists of a cone topped by a spherical cap, both of which are solid). We want to determine the center of inertia (or barycenter $G(x, y, z)$) of this spinner, as well as the moment of inertia J_Δ, with respect to its vertical axis of symmetry Δ. For all calculations, we'll neglect the small cylindrical part at the top of the spinner used to drive it with the fingertips.



(a) Spinner 3D view

(b) Spinner 2D section view

Sin α R

Figure 1: Spinner

1. What are the coordinates x_G and y_G of the spinner's center of inertia? Justify your answer.

2. Using cylindrical coordinates, show that the $z_{G_{\text{cone}}}$ coordinate of the lower part of the spinner, which has the shape of a half-cone (see Figure 1(b)), can be expressed as

$$z_{G_{\text{cone}}} = \frac{1}{M_{\text{cone}}} \int_{z_{\min}}^{z_{\max}} f(z)\, \mathrm{d}z.$$

What is the expression of $f(z)$? Determine $z_{G_{\text{cone}}}$ as a function of α, μ, R and M_{cone}.

– 3. Using a calculation similar to the previous one, show that the $z_{G_{\text{cap}}}$ coordinate of the upper part of the spinner in the shape of a spherical cap as a function of ~~H~~ and R.

$$z_{G_{\text{cap}}} = \frac{1}{M_{\text{cap}}} \cdot \frac{\pi \mu R^4 \cos^4(\alpha)}{4}$$

4. Deduce z_G for the whole spinner as a function of M_{cone}, $z_{G_{\text{cone}}}$, M_{cap} and $z_{G_{\text{cap}}}$.

5. Using spherical coordinates, show that the expression of J_Δ in spherical coordinates can be expressed as

$$J_\Delta = C \int_0^R \int_{\phi_1}^{\phi_2} \rho^n \sin^m(\phi)\, \mathrm{d}\phi \mathrm{d}\rho.$$

Give the expressions for C, the integers n and m, and the values of ϕ_1 and ϕ_2.